THE THEORY OF MEASURE IN ARITHMETICAL SEMI-GROUPS

BY

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PREFACE

The principal object of this study is the isolation of those properties of the asymptotic distribution of the primes in the sequences of all positive integers which remain valid if the latter sequence is replaced either by an arbitrary arithmetical semi-group, or, more generally, by any measurable set of positive integers. By measure is meant relative measure, that is, asymptotic frequency.

The positive results for a semi-group generated by an arbitrary subsequence of the sequence of all primes contain the extension of the prime number theorem to an arbitrary semi-group; although the proof depends on, and the result is actually equivalent to, the prime number theorem. However, the negative results prove the necessity of the approach followed in the case of an arbitrary semi-group.

If a set of positive integers, instead of being a semi-group, is an arbitrary measurable set, the relevant extension of the prime number theorem is still true. This extension supplies a transcendental evaluation of the measure, whenever the latter exists. Corresponding to the lack of a basis of multiplication in this general case, the proof now depends on somewhat more than the prime number theorem.

The problems considered have been suggested by the concluding sections of the monograph cited under [45] in the Bibliography. However, the presentation is self-contained, since even a free use of the results of [45] could have saved only a few preparatory sections at the beginning of Chapter I; sections which are adjusted to the present needs.

I wish to express my indebtedness to Professor E. K. Haviland for his valuable help in seeing the manuscript through the press, and to the Waverly Press for their co-operation.

Herrington Manor, Md., June 1943. AUREL WINTNER

CONTENTS

Chapter I

THE ARITHMETICAL DERIVATIVES

1. If either of two functions, say $f = f(n)$ and $f' = f'(n)$, of the positive integer n is given, the other function is uniquely determined by the condition

$$f(n) = \sum_{d \mid n} f'(d), \qquad (n = 1, 2, \cdots). \tag{1}$$

In fact, this linear transformation of f' into f possesses the unique inversion

$$f'(n) = \sum_{d \mid n} \mu(n/d) f(d), \qquad (n = 1, 2, \cdots). \tag{2}$$

It is understood that the summation index runs through all divisors $d = 1, \cdots, n$ of n, and that $\mu(n)$ denotes the Möbius function.

It is clear from (1) that

$$\sum_{m=1}^{n} f(m) = \sum_{m=1}^{n} \left[\frac{n}{m}\right] f'(m), \tag{3}$$

where the bracket refers to the integral part of the fraction n/m. It is also seen from (1) that, if p is a prime,

$$(4_1) \qquad f(p^k) = \sum_{j=0}^{k} f'(p^j); \qquad\qquad (4_2) \qquad f(1) = f'(1).$$

If the accent defining the transformation, (2), of $f(n)$ into $f'(n)$ is thought of as a symbol of "arithmetical derivation", the function (1) may be denoted by $\int h(n)$, where $h = f'$. This notation generalizes Euler's symbol, $\int (n)$, for the sum of the divisors of n, since (1) becomes this sum when $f'(n) = n$. Correspondingly, if $\tau(n)$ denotes the number of the divisors of n, then

$$\tau'(n) = 1, \tag{5}$$

by (1). Similarly, if $\epsilon(n)$ is defined by

$$\epsilon(1) = 1 \quad \text{and} \quad 0 = \epsilon(2) = \epsilon(3) = \epsilon(4) = \cdots, \tag{6}$$

then (1) shows that the function $f(n) \equiv 1$ belongs to

$$1' = \epsilon(n). \tag{7}$$

According to (2) and (6), the Möbius function may be defined by

$$\epsilon'(n) = \mu(n); \qquad \text{cf. (6)}. \tag{8}$$

If $\phi(n)$ denotes Euler's function, then

$$(7\text{ bis}) \qquad n' = \phi(n); \qquad\qquad (8\text{ bis}) \qquad (\phi(n)/n)' = \mu(n)/n.$$

Finally, it is easily verified from (1), (2) that

(9_1) $\quad (\log n)' = \Lambda(n);$ $\qquad$ (9_2) $\quad \Lambda'(n) = -\mu(n) \log n,$

if $\Lambda(n)$ denotes, as usual, the function which is $\log p$ or 0 according as n is or is not a prime power p^k, where $k = 1, 2, \cdots$.

2. According to the uniqueness theorem of Dirichlet series, the definition (1) is formally equivalent to the identity

$$\sum f(n)/n^s = \zeta(s) \sum f'(n)/n^s, \tag{10}$$

where

$$\sum g(n) = \sum_{n=1}^{\infty} g(n). \tag{11}$$

More specifically, if either of the series $\sum f'(n)/n^s$, $\sum f(n)/n^s$ converges in a half-plane $\sigma > \beta$, where $s = \sigma + it$ and $\beta > 1$, then both of these series converge for $\sigma > \beta$ and satisfy (10). This follows from (1), (2) and from the multiplication theorem of Mertens-Stieltjes, since both $\zeta(s) = \sum 1/n^s$ and $1/\zeta(s) = \sum \mu(n)/n^s$ are absolutely convergent for $\sigma > 1$.

The prime number theorem (Hadamard; de la Vallée-Poussin), i.e. $\sum_{n=1}^{x} \Lambda(n) \sim x$, is equivalent to

$$\sum_{n=1}^{x} \mu(n) = o(x) \tag{12}$$

(Landau) and also to the non-vanishing of $\zeta(s)$ on the line $\sigma = 1$ (Ikehara). Correspondingly, those domains beyond the line $\sigma = 1$ on which the non-vanishing of $\zeta(s)$ has been established lead not only to explicit estimates of the remainder term of the prime number theorem but also to respective improvements of the estimate (12). For instance, even de la Vallée-Poussin's zero-free domain supplies more than that

$$\sum_{n=1}^{x} \mu(n) = o(x/\log^{\lambda} x) \tag{13}$$

holds for *every* fixed index λ.

In the sequel, (13) will repeatedly be needed for *some fixed* $\lambda > 1$. It will be of methodical importance that the existence of a $\lambda > 1$ (and, for that matter, even the existence of a $\lambda > 0$) is not implied by the prime number theorem, that is, by (12), where $\lambda = 0$.

3. For any function $a(n)$, let $M(a)$ denote the mean

$$M(a) = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} a(m), \tag{14}$$

if (14) exists as a *finite* limit. Thus the existence of $M(f)$ means the $(C, 1)$-summability of the series for which $f(n)$ is the n-th partial sum. In particular, this series must then be summable in the sense of Abel to the value $M(f)$, that is, the power series

$$(15) \qquad \sum g(n)r^n \equiv (1 - r) \sum f(n)r^n, \qquad \text{where } f(n) = \sum_{m=1}^{n} g(m),$$

must converge for $r < 1$ and tend to $M(f)$ as $r \to 1$ (Frobenius).

It is known (Hardy) that the proof (though not the wording) of the implication just mentioned also entails that every series which is $(C, 1)$-summable to a value α is summable in the sense of Lambert to the same α. It is understood that a series $\sum h(n)$ is said to be summable in the sense of Lambert if the series

$$H(r) \equiv \sum nh(n)r^n/(1 - r^n)$$

converges for $r < 1$ and $(1 - r)H(r)$ tends to a limit, α, as $r \to 1$.

If (A) and (L) refer to the summation processes of Abel and Lambert respectively, the above implications are $(C, 1) \to (A)$ and $(C, 1) \to (L)$. Both are elementary in nature. It is also an elementary fact that, if a series is (A)-summable to α_1 and (L)-summable to α_2, then $\alpha_1 = \alpha_2$.

What is not an elementary fact is the implication $(L) \to (A)$. This implication contains the prime number theorem. However, the only known proof of $(L) \to (A)$, due to Hardy and Littlewood [16], presupposes more than the prime number theorem, since the proof depends on the existence of a $\lambda > 1$ satisfying (13).

4. It is clear that the definition (1) is formally equivalent to the identity

$$(16) \qquad \sum f(n)r^n = \sum f'(n)r^n/(1 - r^n).$$

More specifically, if either of the series (16) is convergent (hence absolutely convergent) for $r < \theta$, where $0 < \theta < 1$, then both of these series converge for $r < \theta$ and satisfy (16). This leads to the following elementary lemma:

(i) *If $M(f)$ exists and $\sum f'(n)/n$ converges, then $M(f) = \sum f'(n)/n$.*

In fact, if the series $\sum f'(n)/n$ converges, it is (L)-summable to the value $\sum f'(n)/n$. This means, by (16), that the power series (15), where $r < 1$, tends to the limit $\sum f'(n)/n$ as $r \to 1$. Since, if $M(f)$ exists, the limit of the power series (15) as $r \to 1$ is $M(f)$, the assertion (i) follows.

Neither of the assumptions of (i) is contained in the other:

(ii) *The convergence of $\sum f'(n)/n$ is neither necessary nor sufficient for the existence of $M(f)$.*

This is proved by a pair of counter-examples (Wintner [45], pp. 11–13), which will not be reproduced here.

(iii) *The existence of $M(f)$ and the convergence of $\sum f'(n)/n$ are respectively sufficient for the (A)-summability of $\sum f'(n)/n$ and for the (A)-existence of $M(f)$.*

It is understood that by the (A)-existence of $M(f)$ is meant that the power series (15) converges for $r < 1$ and tends to a limit as $r \to 1$. According to (16),

this assumption is identical with the (L)-summability of the series $\sum f'(n)/n$ and is therefore implied by the convergence of $\sum f'(n)/n$. Thus the second of the assertions of (iii) follows elementarily.

Since the existence of $M(f)$ is sufficient for the (A)-existence of $M(f)$, the remaining assertion of (iii) is contained in the first part of the following theorem:

(iv) *The (A)-existence of $M(f)$ is sufficient, though not necessary, for the (A)-summability of $\sum f'(n)/n$.*

According to (15) and (16), the first of the assertions of (iv) is identical with the difficult implication $(L) \rightarrow (A)$, quoted at the end of §3. Correspondingly, all that the second part of (iv) claims is that the implication $(A) \rightarrow (L)$ does not hold. An example to this effect was constructed by Hardy and Littlewood [17].

5. Tauber's own theorem states that a series $\sum a(n)$ converges if and only if it is (A)-summable and such that $M(b)$, where $b(n) = na(n)$, exists *and is zero*. Actually, this italicized proviso is superfluous. In fact, the existence of $\beta = M(b)$ necessitates its (A)-existence, i.e., the relation $(1 - r)\sum b(n)r^n \rightarrow \beta$ as $r \rightarrow 1$, which, if $\beta \neq 0$, implies (after division by $1 - r$ and integration) that $\sum b(n)r^n/n \sim \beta \log (1 - r)^{-1}$ as $r \rightarrow 1$; so that the series $\sum a(n)r^n$, where $a(n) = b(n)/n$, cannot be (A)-summable, if $\beta = M(b)$ exists but is not zero.

If the resulting form of Tauber's theorem is applied to $a(n) = f'(n)/n$, it follows that a series $\sum f'(n)/n$ is convergent if and only if it is (A)-summable and such that $M(f')$ exists. It follows therefore from the formulation (iv) of $(L) \rightarrow (A)$, that the (A)-existence of $M(f)$ implies the convergence or the divergence of $\sum f'(n)/n$ according as $M(f')$ does or does not exist. This alternative contains the following theorem:

(v) *If $M(f)$ exists, then the convergence of $\sum f'(n)/n$ is equivalent to the existence of $M(f')$.*

On the other hand, if $\sum f'(n)/n$ converges, then the existence of $M(f)$ is not equivalent to the existence of $M(f')$. This follows from the second of the negations of (ii), since the convergence of a series $\sum f'(n)/n$ always implies the existence (and, incidentally, the vanishing) of $M(f')$; cf. (17*) below.

In the sequel, (v) will repeatedly be combined with the following elementary fact:

(v bis) *If $M(f')$ exists, then the limit relation*

$$(17) \quad \frac{1}{n}\sum_{m=1}^{n} f(m) - \sum_{m=1}^{n} \frac{f'(m)}{m} \rightarrow (C - 1)M(f') \quad \text{as} \quad n \rightarrow \infty; \quad \text{cf. (14)},$$

where $C = 0.57 \cdots$ is Euler's constant, holds whenever $f'(n) = O_L(1)$.

First, if $a(n)$ is any function for which the n-th partial sums of the series $\sum na(n)$ and $\sum n \mid a(n) \mid$ are $o(n)$ and $O(n)$ respectively, then

$$\sum_{m=1}^{n} ([n/m]m/n - 1)a(m) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

This elementary Tauberian fact is standard (cf., e.g., Hecke [20], p. 202); it is due to Axer. If it is applied to $a(n) = f'(n)/n$, it implies, in view of (3), that

(17) is true if, on the one hand, $M(f')$ exists and vanishes and, on the other hand, the n-th partial sum of $\sum | f'(n) |$ is $O(n)$; in fact, (14) shows that $M(f')$ exists and vanishes if and only if the n-th partial sum of $\sum na(n)$ is $o(n)$ in the case $a(n) = f'(n)/n$. But if $M(f')$ exists and $f'(n) = O_L(1)$, then it is clear from (14) that the n-th partial sum of $\sum | f'(n) |$ must be $O(n)$. Consequently, (17) is true if, on the one hand, $M(f')$ exists and vanishes and, on the other hand, $f'(n) = O_L(1)$. In other words, (v bis) is true in the particular case $M(f') = 0$. It follows therefore from the distributive character of all the operations occurring in (1), (14) and (17), that, in order to complete the proof of (v bis), it is sufficient to ascertain the truth of (17) for the particular function $f(n)$ for which $f'(n) \equiv 1$. But (5) shows that the assertion of (17) then becomes

(17 bis) $$\sum_{m=1}^{n} \tau(m) - n \sum_{m=1}^{n} 1/m = (C - 1)n + o(n);$$

a relation which, according to Dirichlet's elementary estimate in his divisor problem (contained in (91) below), holds even if the $o(n)$ is replaced by $O(n^{\frac{1}{2}})$.

In connection with (v), the rôle of (v bis) is characterized by the fact that the existence, and even the vanishing, of $M(f')$ is a necessary condition for the convergence of $\sum f'(n)/n$. In fact, if the trivial fact (24) below is applied to $a(n) = f'(n)/n$, it follows that

(17*) $$M(f') = 0 \text{ whenever } \sum f'(n)/n \text{ converges.}$$

6. The following theorems will have the same methodical structure as (iii) or (v); in the sense that one of the two assertions of each of the theorems involves the difficult implication $(L) \rightarrow (A)$, while the other assertion of the theorem does not depend on the prime number theorem, and still less on the existence of a $\lambda > 1$ satisfying (13).

In the latter regard, the relative depths of (vi) and (vii) below are substantially distinct, the situation being as follows: While the proof of the general implication $(L) \rightarrow (A)$ of Hardy and Littlewood depends on *more* than the prime number theorem, Wiener (cf. [40], pp. 116–119) has shown that that conditional (that is, "Tauberian") particular case of the unconditional (that is, "Abelian") implication $(L) \rightarrow (A)$ in which the coefficients $f'(n)$ are assumed to be $O_L(1)$ is *equivalent* to the prime number theorem. Correspondingly, the difficult part of (vi) below, being provable under Wiener's (superfluous) Tauberian restriction of the coefficients, is just equivalent to the prime number theorem. On the other hand, the full force of the unconditional implication $(L) \rightarrow (A)$ will be needed for the difficult part of (vii) below; cf. the example in §14.

The elementary facts represented by (v bis) and (17*) obviously imply one part of the following theorem:

(vi) *The existence of $M(f)$ is equivalent to the convergence of $\sum f'(n)/n$, if $f'(n) = O_L(1)$.*

The difficult part of (vi) follows from the corresponding part of (iii), since, according to an elementary Tauberian theorem of Hardy and Littlewood, the

(A)-summability of a series $\sum a_n$ implies its convergence whenever $na(n) = O_L(1)$.

The Tauberian theorem just quoted is known to be a variant of the following one: If $f(n)$ denotes the n-th partial sum of an (A)-summable series, the latter is $(C, 1)$-summable whenever $f(n) = O_L(1)$. This and the elementary part of (iii) clearly imply the elementary part of the following theorem:

(vii) *The existence of* $M(f)$ *is equivalent to the convergence of* $\sum f'(n)/n$, *if* $f(n) = O_L(1)$.

In view of (v), the remaining part of (vii) is equivalent to the statement that, if $f(n) = O_L(1)$, the existence of $M(f)$ implies the existence of $M(f')$. But if $f(n)$ is replaced by $f(n) + c$, where c is a constant, then $M(f)$ is increased by c and $M(f')$ remains unaltered, since $M(c') = cM(1') = 0$, by (7) and (6). Hence, the assumption $f(n) = O_L(1)$ can be reduced to the case $f(n) > 0$. Furthermore, if $M(f')$ exists at all, (17*) shows that $M(f') = 0$. Thus the assertion is that $M(f') = 0$ is implied by the existence of $M(f)$, if $f(n) > 0$. Hence, if $M(f)$ is denoted by α, it is seen from (14) that what is to be proved is the following fact: If $f(n) > 0$, and if there exists a constant, α, such that the $[x]$-th partial sum of $\sum f(n)$ is $\alpha x + o(x)$, then the $[x]$-th partial sum of $\sum f'(n)$ is $o(x)$.

For a fixed x, the latter partial sum is identical with

$$\sum_{n=1}^{x^{\frac{1}{2}}} f(n) \sum_{m=1}^{x/n} \mu(m) + \sum_{n=1}^{x^{\frac{1}{2}}} \mu(n) \sum_{m=1}^{x/n} f(m) - \sum_{n=1}^{x^{\frac{1}{2}}} f(n) \sum_{m=1}^{x^{\frac{1}{2}}} \mu(m),$$

as easily verified from (2) by partial summation (actually, this representation of the $[x]$-th partial sum of $\sum f'(n)$ is simply the case $t = x^{\frac{1}{2}}$ of the general identity following (29) below). Since $1 \leqq m \leqq x/n$ and $1 \leqq n \leqq x^{\frac{1}{2}}$ imply that $x/n \geqq x^{\frac{1}{2}}$, and since the $[y]$-th partial sums of $\sum \mu(m)$ and $\sum f(m)$ are $o(y)$ and $\alpha y + o(y)$, by (12) and by assumption, respectively, it follows, by placing $y = x^{\frac{1}{2}}$, that the $[x]$-th partial sum of $\sum f'(n)$ is of the form

$$\sum_{n=1}^{x^{\frac{1}{2}}} | f(n) | \, o(x^{\frac{1}{2}}) + \sum_{n=1}^{x^{\frac{1}{2}}} \mu(n)\{\alpha x/n + o(x^{\frac{1}{2}})\} + O(x^{\frac{1}{2}})o(x^{\frac{1}{2}}),$$

where the o-terms are uniform in $n(\leqq x^{\frac{1}{2}} \to \infty)$. Since $| f(n) | = f(n)$ by assumption, and since $\mu(n) = O(1)$, this can be written as

$$o(x^{\frac{1}{2}}) \sum_{n=1}^{x^{\frac{1}{2}}} f(n) + \alpha x \sum_{n=1}^{x^{\frac{1}{2}}} \mu(n)/n + o(x^{\frac{1}{2}}) \sum_{n=1}^{x^{\frac{1}{2}}} O(1) + O(x^{\frac{1}{2}})o(x^{\frac{1}{2}}).$$

But this altogether is just $o(x)$, since the $[x^{\frac{1}{2}}]$-th partial sums of $\sum f(n)$ and $\sum \mu(n)/n$ are $\alpha x^{\frac{1}{2}} + o(x^{\frac{1}{2}})$ and $o(1)$ respectively (in fact, all that this $o(1)$ states is that the series $\sum \mu(n)/n$ converges and has the sum 0, a statement which is known to be equivalent to (12); cf. §8 below).

This completes the proof of (vii).

7. In order to avoid interruptions of later considerations, it is convenient to isolate the following elementary fact:

LEMMA. *If* $M(a)$ *and* $M(b)$ *exist and are* 0 *for two functions,* $a = a(n)$ *and*

$b = b(n)$, and if $a(n) = O(1)$ and

$$\sum_{m=1}^{n} b(m) = O(n/\log^{\lambda} n) \quad \text{for some} \quad \lambda > 1, \tag{18}$$

then $M(c)$ exists and is 0 for the function $c(n)$ defined by

$$c(n) = \sum_{d\mid n} a(d)b(n/d), \quad \text{i.e.,} \quad c(n) = \sum_{d\mid n} b(d)a(n/d),$$

where the summation index d (or the quotient n/d) runs through all divisors of n.

This Lemma will be proved by placing

$$A(x) = \sum_{m=1}^{x} a(m), \qquad B(x) = \sum_{m=1}^{x} b(m), \qquad C(x) = \sum_{m=1}^{x} c(m), \tag{19}$$

where $\sum_{m=1}^{x}$ denotes $\sum_{m=1}^{[x]}$, and using the identity

$$C(x) = \sum_{n=1}^{t} a(n)B(x/n) + \sum_{n=1}^{x/t} b(n)A(x/n) - A(t)B(x/t), \tag{20}$$

where t is any number satisfying $1 \leqq t \leqq x$, i.e., $t \geqq 1$ and $x/t \geqq 1$. The identity (20) is readily verified from (19) and (18), if a "partial summation" is applied in Dirichlet's fashion (cf. his enumeration of lattice points).

It is clear from (14) and (19) that the assumptions of the Lemma are $A(x) = o(x)$, $a(n) = O(1)$ and $B(x) = O(x/\log^{\lambda} x)$, where $\lambda > 1$, and that the assertion of the Lemma is that these assumptions imply the estimate $C(x) = o(x)$, as $x \to \infty$. Consequently, it is sufficient to show that, if the assumptions just mentioned are satisfied, then all three terms on the right of (20) become $o(x)$ for *some* function $t = t(x)$, where $1 \leqq t \leqq x$ and $x \to \infty$.

Since $A(x) = o(x)$, there exists a function $\epsilon = \epsilon(x)$ satisfying $|A(x)| < x\epsilon(x)$ and $\epsilon(x) \to 0$ as $x \to \infty$, and it is clear that this $\epsilon(x)$ can be chosen so as to be a decreasing function of x. By choosing $\epsilon(x)$ large enough for every x, it can in addition be assumed that $x\epsilon(x) \to \infty$ as $x \to \infty$. Then, if $t = x\epsilon(x)$, the assumption $1 \leqq t \leqq x$ of (20) is satisfied for every large x. Hence, it is sufficient to show that all three terms on the right of (20) are $o(x)$, if $t = x\epsilon(x)$ and $A(x) = o(x)$, $a(n) = O(1)$, $B(x) = O(x/\log^{\lambda}x)$, where $\lambda > 1$.

First, $A(x) = o(x)$ and $B(x) = O(x)/\log^{\lambda}x$ imply that the third term is

$$A(t)\, B(x/t) = o(t)\, O(x/t)/\log^{\lambda}(x/t) = o(x)/\,|\log \epsilon(x)|^{\lambda} = o(x)\, o(1),$$

since $t = x\epsilon(x)$ and $\epsilon(x) \to 0$ (only $\lambda > 0$ is used here).

Next, since $|A(x)| < x\epsilon(x)$ implies that $|A(x/n)| < x\epsilon(x/n)/n$, and since the function $\epsilon(x)$ was chosen to be decreasing, $|A(x/n)| < x\epsilon(x)/n$. On the other hand, $B(x) = O(x/\log^{\lambda}x)$ implies that $B(x) - B(x-1) = O(x/\log^{\lambda}x)$, which means, by (19), that $b(n) = O(n/\log^{\lambda}n)$. Hence, the second term on the right of (20) is

$$\sum_{n=2}^{x/t} O(n/\log^{\lambda} n)x\epsilon(x)/n = x\epsilon(x)\, O\int_{2}^{x/t} (\log n)^{-\lambda}\, dn,$$

i.e., $x\epsilon(x)O(x/t)/\log^\lambda(x/t)$, or, since $t = x\epsilon(x)$, simply

$$x\epsilon(x)O(1/\epsilon(x))/\mid\log\epsilon(x)\mid^\lambda = O(x)/\mid\log\epsilon(x)\mid^\lambda,$$

which is $O(x)o(1)$, since $\epsilon(x) \to 0$ and $\lambda > 0$.

Finally, $a(n) = O(1)$ and $B(x) = O(x)/\log^\lambda x$ imply that the first term on the right of (20) is

$$\sum_{n=1}^{t} O(1)(x/n)/\log^\lambda(x/n) = O(x)\sum_{n=1}^{t}\{n\log^\lambda(x/n)\}^{-1}$$

or, since $t = x\epsilon(x)$,

$$O(x)\int_1^{x\epsilon(x)}\{n\log^\lambda(x/n)\}^{-1}\,dn = O(x)\int_{1/\epsilon(x)}^{x}\{u\log^\lambda u\}^{-1}\,du,$$

where $u = x/n$ (for a fixed x). But the last integral is $o(1)$ as $x \to \infty$, since $\epsilon(x) \to 0$ and $\lambda > 1$.

This completes the proof of the Lemma.

8. It is seen from (2) that the function $c(n)$ defined by (18) is $f'(n)$, if $a(n) = \mu(n)$ and $b(n) = f(n)$. But then $a(n) = O(1)$ is trivial from $\mid\mu(n)\mid \leqq 1$. On the other hand, the existence of $M(a) = 0$ for $a(n) = \mu(n)$ is equivalent to (12). Consequently, the elementary nature of the Lemma of §7 contains the following fact:

If $f(n)$ is any function satisfying

$$\sum_{m=1}^{n} f(m) = O(n/\log^\lambda n) \quad \text{for some} \quad \lambda > 1, \tag{21}$$

then the existence (and the vanishing) of $M(f')$ is an elementary consequence of the formulation (12) of the prime number theorem.

It may be mentioned that this fact entails the elementary nature of the passage from (12) to the standard formulation, $\Lambda(1) + \cdots + \Lambda(n) \sim n$, of the prime number theorem. In order to see this, let C be a constant (which afterwards will be chosen to be Euler's constant), and let

$$f(n) = \log n - \tau(n) + 2C. \tag{22}$$

Then, the transformation (2) of f into f' being distributive, (9_1), (5) and (7) show that $f'(n) = \Lambda(n) - 1 + 2C\epsilon(n)$. According to (6), the existence and the vanishing of $M(f')$ for this $f'(n)$ is equivalent to $M(\Lambda - 1) = 0$ or $M(\Lambda) = 1$, i.e., to the standard formulation of the prime number theorem. Hence all that remains to be ascertained is that (21) is satisfied by (22) for elementary reasons. But this is implied by Dirichlet's $O(n^{\frac{1}{2}})$-estimate, mentioned after (17 bis).

Incidentally, (6) and (8) show that, if $f(n) = \epsilon(n)$, then (21) is satisfied and $M(f') = 0$ becomes precisely (12).

If $f(n)$ is the function defined by $f'(n) = \mu(n)$, then $f'(n) = O(1)$. Furthermore, (6) and (8) show that all partial sums of $\sum f(n)$ are 1 in this case, and that (12) is equivalent to the existence and the vanishing of $M(f')$. Hence, if the

elementary Axerian lemma, (v bis), is applied to $f'(n) = \mu(n)$, it follows that the convergence of the series $\sum \mu(n)/n = 0$ is an elementary consequence of the formulation (12) of the prime number theorem. The converse inference, that leading from the convergence of $\sum \mu(n)/n$ to (12), is contained in Kronecker's remark, according to which

$$\sum_{m=1}^{n} ma(m) = o(n) \quad \text{whenever} \quad \sum a(n) \quad \text{converges.} \tag{24}$$

All of this is well-known, of course.

It is now easy to see that (vii), in contrast with (vi), contains *more* than the prime number theorem itself, i.e., than (12). First, (6) and (8) show that (vii) contains the convergence of $\sum \mu(n)/n$ and therefore the existence of $M(\Lambda)$. Since $\Lambda(n) \geqq 0$, it follows that (vii) contains the convergence of the series $\sum f'(n)/n$ when $f'(n)$ is the function (9_2). But a partial summation shows that the convergence of this series contains the case $\lambda = 1$ of estimate (13). This is only slightly less than what has been used in the proof of (vii), namely the case $\lambda = 1 + \epsilon$ of (13) for *some* $\epsilon > 0$.

9. Since the series $\sum \mu(n)/n$ converges to the sum 0, a partial summation transforms (13) into

$$\sum_{n=1}^{x} \mu(n)/n = o(1/\log^{\lambda} x). \tag{25}$$

Although (13), hence (25), holds for *every* $\lambda > 0$, it will again be sufficient to use the existence of *some* $\lambda > 1$.

According to (14), the existence of $M(f)$ is equivalent to the existence of a number, $\alpha = M(f)$, satisfying the first of the two conditions

$$(26_1) \quad \sum_{m=1}^{n} \{f(m) - \alpha\} = o(n); \qquad (26_2) \quad \sum_{m=1}^{n} |f(m) - \alpha| = o(n),$$

which is implied by, but does not imply, the second. Correspondingly, $M(f)$ may be said to exist *absolutely* if there exists a number, $\alpha = M(f)$, satisfying (26_2). Then a criterion depending on (25) can be formulated as follows:

(viii) *The absolute existence of $M(f)$ implies the convergence of $\sum f'(n)/n$.*

In view of (iv), (iii) and (ii), it appears to be of interest that (viii) has the following dual:

(viii bis) *The absolute convergence of $\sum f'(n)/n$ implies the existence of $M(f)$.*

However, while (viii) depends on more than the prime number theorem, (viii bis) is a triviality. In fact, since $0 \leqq x - [x] < 1$, it is clear from (3) that

$$\left| \sum_{m=1}^{n} f(m)/n - \sum_{m=1}^{n} f'(m)/m \right| \leqq \sum_{m=1}^{n} |f'(m)|/n. \tag{27}$$

On the other hand, if (24) is applied to $a(n) = |f'(n)|/n$, it is seen that the numerator of the quotient on the right of (27) is $o(n)$, i.e., that this quotient tends to 0, whenever the assumption, $\sum |f'(n)|/n < \infty$, of (viii bis) is satisfied. Hence the assertion of (viii bis) follows from the inequality (27).

Actually, the trivial assertion (viii bis) is the limiting case $\lambda = 1$ of the following criterion:

(viii*) *The existence of* $M(f)$ *is equivalent to the convergence of* $\sum f'(n)/n$, *if* $\sum |f'(n)|^{\lambda}/n < \infty$ *for some* $\lambda > 1$.

The assertion of (viii*) is that the expression on the left of (27) tends to 0 as $n \to \infty$, if $\sum | m^{-1/\lambda} f'(m) |^{\lambda} < \infty$. But (3) shows that the λ-th power of the expression on the left of (27) is

$$\left| \sum_{m=1}^{n} (1/m - [n/m]/n) f'(m) \right|^{\lambda}$$
$$\leqq \left\{ \sum_{m=1}^{n} | m^{1/\lambda}(1/m - [n/m]/n) |^{\lambda/(\lambda-1)} \right\}^{\lambda-1} \sum_{m=1}^{n} | m^{-1/\lambda} f'(m) |^{\lambda}$$

by Hölder's inequality. Since the contribution of the range $1 \leqq m \leqq k$ to the first sum, $\{\ \}$, on the right of this inequality tends to 0 when k is fixed and $n \to \infty$, it follows that, in order to prove (viii*), it is sufficient to ascertain that the sum $\{\ \}$, belonging to the full range $1 \leqq m \leqq n$, remains bounded as $n \to \infty$. But it is readily verified that the sum $\{\ \}$ is identical with the arithmetical mean of the n values attained by the $\lambda/(\lambda - 1)$-th power of the function $t^{1/\lambda}(t^{-1} - [t^{-1}])$, where $0 < t \leqq 1$, at the points $t = m/n$, where $m = 1, \cdots, n$. Since this function of t is bounded, the proof of (viii*) is complete.

If $f(n) = 1$ for every n, then $M(f)$ exists absolutely and $\sum f'(n)/n$ becomes $1 + 0 + 0 + \cdots$, by (7) and (6); so that the assertion of (viii) is certainly true in this case. It follows therefore from the distributive character of the connection (2) between f and f', that it is sufficient to prove (viii) for the case $M(f) = 0$. Then the assumption (26_2), where $\alpha = M(f)$, means that the $[x]$-th partial sum of $\sum |f(n)|$ is $o(x)$. Thus the $[x]$-th partial sum of $\sum f'(n)$, represented by the second formula line of the proof of (vii), is of the form

$$o(x^{\frac{1}{2}})o(x^{\frac{1}{2}}) + \sum_{n=1}^{x^{\frac{1}{2}}} \mu(n)\{\alpha x/n + o(x^{\frac{1}{2}})\} + O(x^{\frac{1}{2}})o(x^{\frac{1}{2}}).$$

It follows therefore from $\alpha = 0$ and $\mu(n) = O(1)$ that the $[x]$-th partial sum of $\sum f'(n)$ is $o(x)$. This means, by (14), that $M(f')$ exists (and is 0). Hence, the assertion of (viii) follows from (v).

It will be seen in §15 that (vii) does not imply (viii). That (viii) does not imply (vii) is shown by the function $f(n)$ which is 0 unless n is a square, while $f(n^2) = (-1)^n n^{\frac{1}{2}}$. Then $\sum f(n)/n$ converges absolutely. Hence, if (10) is multiplied by $\sum \mu(n)/n^s = 1/\zeta(s)$, it follows from the convergence of $\sum \mu(n)/n$ and from the multiplication theorem of Mertens-Stieltjes, that $\sum f'(n)/n$ is convergent. Furthermore, the absolute convergence of $\sum f(n)/n$ entails, by the case $a(n) = |f(n)|/n$ of (24), the absolute existence of $M(f) = 0$. However, $f(n) = O_L(1)$ is not satisfied, since $f(n^2) = (-1)^n n^{\frac{1}{2}}$.

10. Since the assumption of (viii) is (26_2), where $\alpha = M(f)$, the following criterion is of the same type as (viii).

(ix) *If $M(f)$ exists so strongly that*

$$\sum_{m=1}^{x} f(m)/x = M(f) + o(1/\log^{\lambda} x) \quad \text{for some} \quad \lambda > 1, \tag{28}$$

then $\sum f'(n)/n$ is convergent.

For the same reasons as in the proof of (viii), it can be assumed without loss of generality that $M(f) = 0$. Then (28) means that (21) is satisfied. Hence the italicized result of §8 supplies the existence of $M(f')$. Consequently, the assertion of (ix) follows from (v).

If (18) is identified with (1), so that $a(n) = 1$ for every n, then, from (19) and (20),

$$\sum_{n=1}^{x} f(n) = \sum_{n=1}^{t} \sum_{m=1}^{x/n} f'(m) + \sum_{n=1}^{x/t} f'(n)[x/n] - [t] \sum_{n=1}^{x/t} f'(n), \tag{29}$$

since $A(x) = [x]$. Similarly, if (18) is identified with (2),

$$\sum_{n=1}^{x} f'(n) = \sum_{n=1}^{t} \mu(n) \sum_{m=1}^{x/n} f(m) + \sum_{n=1}^{x/t} f(n) \sum_{m=1}^{x/n} \mu(m) - \sum_{m=1}^{t} \mu(m) \sum_{n=1}^{x/t} f(n).$$

As an application, consider (for later reference) the drastic case of those functions $f(n)$ for which the partial sums of $\sum f'(n)$ are bounded. Then, if t is chosen to be $x^{\frac{1}{2}}$ in (29),

$$\sum_{n=1}^{x} f(n) = \sum_{n=1}^{x^{\frac{1}{2}}} O(1) + \sum_{n=1}^{x^{\frac{1}{2}}} f'(n)[x/n] - x^{\frac{1}{2}}O(1),$$

and so, since $|f'(n)([x/n] - x/n)| \leqq |f'(n)| = O(1)$,

$$\sum_{n=1}^{x} f(n) = O(x^{\frac{1}{2}}) + \sum_{n=1}^{x^{\frac{1}{2}}} f'(n)x/n + \sum_{n=1}^{x^{\frac{1}{2}}} O(1) = x \sum_{n=1}^{x^{\frac{1}{2}}} f'(n)/n + O(x^{\frac{1}{2}}).$$

On the other hand, a partial summation shows that, since the partial sums of $\sum f'(n)$ are bounded, the $[x]$-th partial sum of $\sum f'(n)/n$ tends to a limit, $\sum f'(n)/n$, in such a way that the remainder term is $O(x^{-\frac{1}{2}})$. Hence the preceding formula line implies that, *if the partial sums of $\sum f'(n)$ are bounded, then $M(f)$ exists and $r(n) = O(n^{\frac{1}{2}})$, where*

$$r(n) = \sum_{m=1}^{n} f(m) - M(f)n. \tag{30}$$

Incidentally, it is clear from the proof that, if the partial sums of $\sum f'(n)$, instead of being just bounded, tend to a limit, i.e., *if $\sum f'(n)$ is a convergent series, then $r(n) = o(n^{\frac{1}{2}})$.*

11. If the constant (14) is interpreted as the "mean motion", i.e., the linear function $M(f)n$ as the "secular component", of the sum function $f(1) + \cdots + f(n)$, then there arises the question as to conditions under which either the above estimates of the "oscillating" remainder function, (30), can be improved to $r(n) = O(1)$ or $r(n)$ is really such as to admit an anharmonic analysis (cf. Wint-

ner [42]). It will be shown that a condition sufficient for both of these behaviors results if the assumption of the last italicized statement is replaced by the *absolute* convergence of $\sum f'(n)$. In fact, *if* $\sum f'(n)$ *is absolutely convergent, then* $r(n)$ *is a bounded function and is almost-periodic* (B) in the sense of Besicovitch.* (Needless to say, neither of these properties of an arbitrary function $r(n)$ is implied by the other).

First, the absolute convergence of $\sum f'(m)$ implies that the functions $s_1(n)$, $s_2(n), \cdots$ defined by

$$(31) \qquad s_k(n) = \sum_{m=1}^{k} \{[n/m] - n/m\} f'(m)$$

are uniformly bounded. In fact, the absolute value of the m-th term of (31) is at most $|f'(m)|$, since $|[x] - x| < 1$. On the other hand, from (31) and (3),

$$(30\text{ bis}) \qquad s_n(n) = \sum_{m=1}^{n} f(m) - n \sum_{m=1}^{n} f'(m)/m.$$

Hence, from (30), where $M(f) = \sum_{m=1}^{\infty} f'(m)/m$ in view of (i), §4,

$$|s_n(n) - r(n)| \leqq n \sum_{m=n+1}^{\infty} |f'(m)|/m.$$

Since $1/m$ is less than $1/n$ in the last sum, it follows from the absolute convergence of $\sum f'(m)$ that $s_n(n) - r(n) \to 0$ as $n \to \infty$. This, when combined with the uniform boundedness of the functions $s_1(n), s_2(n), \cdots$, proves that $r(n)$ is a bounded function.

Next, since $[x]$ denotes the integral part of x, the coefficient, $\{\ \}$, of $f'(m)$ in (31) is a periodic function of n (with m as a period). Hence, the finite sum (31) is a periodic function of n (with a period depending on the subscript, k). Since $|[x] - x| < 1$, it is also clear from (31) that, if $n > k$,

$$|s_n(n) - s_k(n)| \leqq \sum_{m=k+1}^{n} |f'(m)| \leqq \epsilon_k, \qquad \text{where} \quad \epsilon_k = \sum_{m=k+1}^{\infty} |f'(m)|.$$

But $\epsilon_k \to 0$ as $k \to \infty$, since $\sum f'(m)$ is supposed to be absolutely convergent. Since $|s_n(n) - s_k(n)| \leqq \epsilon_k$ for every $n > k$, it follows that the functions $s_1(n)$, $s_2(n), \cdots$ tend to the function $s_n(n)$ in the mean of the Besicovitch space. Thus the periodicity of each of the functions $s_1(n), s_2(n), \cdots$ implies the almost-periodicity (B) of $s_n(n)$. This proves the almost-periodicity (B) of $r(n)$, since, as shown before, $s_n(n) - r(n) \to 0$ as $n \to \infty$.

* This becomes of particular interest if it is compared with the fact that the same condition of absolute convergence also implies the uniform almost-periodicity of $f(n)$ itself (in this regard, cf. Wintner [45], p. 34). In fact, the uniform almost-periodicity of a function $f(n)$ is in itself insufficient (and unnecessary) for the boundedness and/or the almost-periodicity (B) of the reduced sum function (30), or "integral", of $f(n)$.

12. The considerations at the end of §10 were based on the choice $t = x^{\frac{1}{2}}$ in the identity (29). If the same choice is made in the identity following (29), it is seen from $|\mu(n)| \leqq 1$ that

$$(32)\qquad \left|\sum_{n=1}^{x} f'(n)\right| \leqq \sum_{n=1}^{x^{\frac{1}{2}}}\left|\sum_{m=1}^{x/n} f(m)\right| + \sum_{n=1}^{x^{\frac{1}{2}}} |f(n)|\, \beta(x/n) + \beta(x^{\frac{1}{2}}) \left|\sum_{n=1}^{x^{\frac{1}{2}}} f(n)\right|,$$

where $\beta(x)$ is an abbreviation for $\left|\sum_{m=1}^{x} \mu(m)\right|$.

As pointed out before (12), the prime number theorem is equivalent to $\beta(x) = o(x)$; cf. §8. But the difficult assertions of (vi) and (vii) can respectively be thought of as *invariant* formulations and implications of the prime number theorem and of its refinement $\beta(x) = o(x/\log^{\lambda} x)$, where $\lambda > 1$; formulations and implications which transcribe (12) or (13) from the case of the *single* pair (6), (8) to the case of *arbitrary* mates (1), (2). Clearly, the possibility of such a transcription is just a manifestation of Toeplitz's general norm-principle in the theory of linear transformations of sequences; the relation (2) being a *fixed* linear transformation of an arbitrary sequence $f'(1), f'(2), \cdots$ into an arbitrary sequence $f(1), f(2), \cdots$. Correspondingly, the counter-examples proving (ii) have been constructed (loc. cit.) precisely on the basis of Toeplitz's norm-principle.

Thus (32) suggests an application of the norm-principle in the direction which is the reverse of that leading to (vi) or (vii). In other words, instead of proving that the function $\beta(x)$ belonging to the *fixed* mates (6), (8) fails to satisfy a hypothetical estimate, it is sufficient to show that the estimate in question leads, via (32), to estimates which ought to hold for *arbitrary* mates $f(n), f'(n)$ but turn out to be violated by *suitable* mates $f(n), f'(n)$.

As an illustration, consider the hypothesis which Mertens [31] formulated as follows: $\beta(x) \leqq x^{\frac{1}{2}}$ for every x. Although v. Sterneck collected considerable numerical evidence for the sharper statement $\beta(x) < \frac{1}{2}x^{\frac{1}{2}}$ when x is sufficiently large, even the truth of the weaker statement $\beta(x) = O(x^{\frac{1}{2}})$ appears to be quite doubtful since Littlewood's work on the remainder term of the formulation $\pi(x) \sim x/\log x$ of the prime number theorem (on the other hand, Littlewood has shown that Riemann's hypothesis is equivalent to $\beta(x) = O(x^{\frac{1}{2}+\epsilon})$, where $\epsilon > 0$ is arbitrarily fixed). However, the hypothesis $\beta(x) = O(x^{\frac{1}{2}})$ has never been refuted. If it is false, it can be disproved by the procedure described above.

In fact, if $\beta(x) = O(x^{\frac{1}{2}})$, then (32) shows that

$$(?)\qquad \sum_{n=1}^{x} f'(n) = O(x^{\frac{1}{2}}) \sum_{n=1}^{x} |f(n)|/n^{\frac{1}{2}} \quad \textit{whenever} \quad \sum_{n=1}^{x} f(n) = O(1).$$

Conversely, if (?) is a true assertion, then $\beta(x) = O(x^{\frac{1}{2}})$ follows by choosing $f'(n) = \mu(n)$, since then $f(2) = f(3) = \cdots = 0$, by (8) and (6). Thus $\beta(x) = O(x^{\frac{1}{2}})$ is true if and only if not a *single* $f(n)$ violates (?). In particular, $\beta(x) = O(x^{\frac{1}{2}})$ is true if and only if the $[x]$-th partial sum of $\sum f'(n)$ is $O(x^{\frac{1}{2}})$ in case of

any function for which the partial sums of both series $\sum f(n)$, $\sum |f(n)|/n^{\frac{1}{2}}$ are bounded.

It will be observed that, in a certain sense, this would amount to the truth of a "dual" of the elementary fact italicized before (30). Correspondingly, all of this could be derived from an O-variant of Stieltjes' convergence theorem for ordinary Dirichlet products.

13. Without an indirect reference to Littlewood's iterated logarithms on which his negative result on the remainder term of $\pi(x) \sim x/\log x$ depends, the unconvincing nature of a numerical evidence concerning the behavior of the function $\mu(n)$ may be illustrated elementarily, as follows:

For $n = 1, 2, \cdots$, let i_n denote the non-negative integer characterized by the property that $\mu(m)$ is 0 whenever $n \leqq m \leqq n + i_n - 1$ but is not 0 for $m = n + i_n$. In other words, i_n is the number of all those consecutive integers following and including n none of which is square-free.

Is $i_n = O(1)$ as $n \to \infty$? Mertens [31] conjectured his hypothesis $\beta(x) \leqq x^{\frac{1}{2}}$ from a tabulation of the values of $\mu(n)$ up to $n = 10^4$, and an inspection of his table shows that $i_n \leqq 5$ in this range. There are up to 10^4 five values of n satisfying $i_n = 5$, and the last of them occurs for $n = 5046$, or, roughly, as early as the middle of the range; the first of them occurs when $n = 844$. In addition, $i_n = 4$ first occurs when $n = 242$, and the next time when $n = 3174$ (there are three more such values of n up to 10^4). But it would be unreasonable to expect that i_n has a finite maximum. The following complete induction will prove $\limsup i_n = \infty$ in a manner exhibiting the genesis of that "iterated logarithm" to which the *slow* appearance of large values of i_n is due.

For a fixed non-negative integer i, suppose that $i_n = i$ is satisfied by some n. This means the existence of a $k = k_i$ for which none of the i consecutive integers $k, k + 1, \cdots, k + i - 1$ is square-free. But if $j = j_i$ is *any* common multiple of these i integers, and if m is *any* positive integer, then each of the i consecutive integers

$$(33) \qquad mj + k,\ mj + k + 1, \cdots, mj + k + i - 1$$

is divisible by some of the i integers $k, k + 1, \cdots, k + i - 1$. Since none of the latter is square-free, none of the integers (33) is. Hence, in order to complete the induction by proving that $i_n = i + 1$ is satisfied by some n, it is sufficient to show that, if the common multiple j and the integer m are suitably chosen, then the integer following the sequence (33), that is, the integer $mj + k + i$, is not square-free.

To this end, choose j so that it is a multiple of $k + i$. Then $h = j/(k + i)$ is an integer satisfying $mj + k + i = (mh + 1)(k + i)$. Hence, $mj + k + i$ cannot be square-free if $mh + 1$ is not square-free. Since no restriction was placed on the choice of m in (33), all that remains to be ascertained is that, if h is a given integer, there exists an $m = m_h$ for which $mh + 1$ is not square-free. But this follows by choosing $m = h + 2$, since $mh + 1$ then is the square of $h + 1$.

Chapter II

THE ZETA-FUNCTIONS

14. If S is any set of distinct positive integers, let $S(n)$ denote its characteristic function, that is, the function which is 1 or 0 according as n is or is not in S; so that $S(n) = \frac{1}{2} \pm \frac{1}{2}$. Thus, from (2),

$$S'(n) = \sum_{d \mid n}^{S} \mu(n/d), \tag{34}$$

where the summation index, d, runs through those divisors ($\geqq 1$) of n which are contained in S.

For instance, (7) and (8) respectively state that $S'(n) = 0$ for $n > 1$ but $S'(1) = 1$, if S consists of all positive integers, and that $S'(n) = \mu(n)$ for every n, if S consists of the single integer 1; so that $| S'(n) | \leqq 1$ holds for every n in both of these extreme cases. In the general case, the sum (34) consists of not more than $\tau(n)$ non-vanishing terms, each of which is ± 1, if $\tau(n)$ denotes the number of all divisors of n. However, the resulting absolute estimate, $| S'(n) | \leqq \tau(n)$, of a function $S'(n)$ belonging to an arbitrary set, S, appears to be too rough. Thus there arises the question as to two absolute orders of magnitude, say $\alpha(n)$ and α_n, defined as follows:* While $\alpha(n)$ is the least bound satisfying $| S'(n) | \leqq \alpha(n)$ for every set S when n is fixed, the estimate $S'(n) = O(\alpha_n)$ holds, and cannot be improved, when S is fixed and $n \to \infty$. All that is obvious is that the order of $\alpha(n)$ cannot be lower than that of α_n. On the other hand, it will be verified below that the order of α_n cannot be lower than that of $\nu(n)$, where $\nu(n)$ denotes the number of the distinct prime divisors of n. Consequently, the orders of both $\alpha(n)$ and α_n are somewhere between those of $\nu(n)$ and $\tau(n)$. The extent of these limitations is seen from the elementary fact (Wigert-Ramanujan) according to which the least monotone majorant not only of $\nu(n)$ but also of $\log \tau(n)$ is asymptotically proportional to $\log n/\log \log n$. In this connection, cf. Besicovitch [1].

In order to verify the above-mentioned lower estimate of α_n, it is sufficient to exhibit a set S for which $| S'(n) | = \nu(n)$ holds whenever n is square-free. But such a set results by choosing S so as to consist of all powers, $p, p^2, \cdots$ of all primes p (Kluyver), since $S'(n)$ then is $-\nu(n)\mu(n)$ for every n. In fact, if n is either 1 or not square-free, then either the first or the second factor of the product $-\nu(n)\mu(n)$ is 0; while the definition of S and the identity (34) show that $S'(n)$ is 0. In the remaining case, n is a product of distinct primes and so, since S consists of all prime powers $p, p^2, \cdots$, the sum (34) becomes

$$S'(n) = \sum_{p \mid n} \mu(n/p). \tag{35}$$

* Another pair of questions results if the sets admitted are required to be measurable in the sense of §15. The lower limitation obtained below is based on a measurable set.

By the definition of $\nu(n)$, the sum (35) has $\nu(n)$ terms. Furthermore, p being a prime divisor of a square-free n, the quotient n/p is the product of $\nu(n) - 1$ distinct primes. Since each of the latter contributes -1 to the factorization of $\mu(n/p)$, and since the $\nu(n)$-th power of -1 is $\mu(n)$ when n is square-free, it follows that each of the $\nu(n)$ terms of the sum (35) is $-\mu(n)$; so that $S'(n) = -\nu(n)\mu(n)$.

Incidentally, (35) becomes correct for every n, if S, instead of being chosen as above, consists only of all primes.

It may be mentioned that the function $S'(n)$ belonging to an arbitrary set S can attain only the values ± 1 and 0, if n is a prime power, $n = p^k$. This is seen by writing (4_1) in the form $f'(p^k) = f(p^k) - f(p^{k-1})$ and observing that a characteristic function, $f(n) = S(n)$, can attain only the values 1 and 0.

15. If the limit (14) exists for the characteristic function, $a(n) = S(n)$, of a set, S, of positive integers, then S will be called measurable, and $M(S)$ its measure. Thus $0 \leqq M(S) \leqq 1$, since $0 \leqq S(n) \leqq 1$. Needless to say, this measure is a "relative" measure and cannot, therefore, possess the fundamental properties of a Lebesgue measure. For example, if $S_1, S_2, \cdots$ are the sets consisting of the single integer representing the first, second, $\cdots$ element in a set, S, which does not have the measure 0, then $S_1, S_2, \cdots$ are mutually disjoint, measurable sets possessing a logical sum which is not measurable or has a measure distinct from the sum ($= 0$) of the measures of the sets $S_1, S_2, \cdots$, according as S is not or is measurable. It is also easy to see that, if *two* measurable sets are not disjoint, then neither their logical sum nor their logical product need be measurable.

Since $0 \leqq S(n) \leqq 1$, the following fundamental fact is contained in (vii):

THEOREM. *A set S is measurable if and only if the series $\sum S'(n)/n$ converges; in which case the series represents the measure, $M(S)$, of S.*

The latter identity, which supplies an *Eratosthenian evaluation of any measure*, follows from the elementary fact (i), §4.

On the other hand, not even (vi) could have sufficed for the proof of the convergence of $\sum S'(n)/n$ in case of an arbitrary measurable set S. Actually, this negation remains valid if S is restricted to be of measure 0. In fact, if S consists of all prime powers, then S is of measure 0, since there are only $o(x)$ prime powers not exceeding x; although the assumption, $S'(n) = O_L(1)$, of (vi) is violated, since $S'(n) = -\nu(n)\mu(n)$, by §14. Thus the Theorem could not have been proved within the *Tauberian* frame-work of the prime number theorem (cf. the comments made at the beginning of §6).

In the example just mentioned, $M(S)$ is 0, which, since $S(n) \geqq 0$, implies that $M(S)$ exists absolutely; cf. the definition (26_2), where $f(n) = S(n)$ and $\alpha = M(S)$. However, since $S(n)$ can attain only the values 1 and 0, it is easily verified that, if S is measurable at all, $M(S)$ exists absolutely if and only if its value is either 1 or 0. Consequently, (viii) could not have sufficed for the proof of the Theorem,

although the deduction of (viii) has involved a refinement, (13), of the prime number theorem, (12).

That the Theorem contains the prime number theorem, is seen by choosing S to be the set consisting of the single number 1. In fact, (6) and (8) show that the assertion of the Theorem then becomes the convergence of $\sum \mu(n)/n$, which in view of (24) implies (and, as explained before (24), is incidentally implied by) the formulation (12) of the prime number theorem.

REMARK. *If* $\int S(n)$ *denotes* the number of those divisors of n which are contained in a set, S, and if S is measurable, then the n-th partial sum of the series* $\sum \int S(n)$ *is asymptotically proportional to n* $\log n$, *with the measure,* $M(S)$, *as factor of proportionality; in fact,*

$$(*) \qquad \frac{1}{n}\sum_{m=1}^{n}\int S(m) - \sum_{m=1}^{n}\frac{S(m)}{m} \to (C - 1)M(S),$$

where C *is Euler's constant and* $S(n)$ *denotes the characteristic function of S.*

It is understood that the asymptotic proportionality means the estimate $o(n \log n)$, if $M(S) = 0$. Whether $M(S) = 0$ or $M(S) > 0$, the asymptotic proportionality is a much weaker statement than (*). In fact, it results if first (*) is divided by $\log n$ and then use is made of the Abelian fact that the n-th partial sum of a series $\sum S(n)/n$ is $\alpha \log n + o(\log n)$ whenever that of the series $\sum S(n)$ is $\alpha n + o(n)$, i.e., whenever $\alpha = M(S)$ exists.

As to (*), it is sufficient to observe that both assumptions of (v bis) are satisfied; so that (*) follows from (1) and (17).

The Remark supplies for the evaluation of the measure of an arbitrary measurable set a rule which represents a *dual* of the evaluation supplied by the Theorem but, in contrast with the latter, is elementary in nature.

16. An arbitrary (that is, not necessarily characteristic) function, $f(n)$, is called multiplicative if it does not vanish for at least one n and possesses the factorization $f(mn) = f(m)f(n)$ when m and n are relatively prime. This implies that

$$(36) \qquad f(1) = 1, \qquad \text{and so} \quad f'(1) = 1,$$

by (4_2). Clearly, a multiplicative function is uniquely determined for every n by an (arbitrary) assignment of its values for $n = p^k$, where p and k run through all primes and all positive integers respectively. Correspondingly, since both functions (6), (8) are multiplicative, it is readily seen from (1), (2) that

$$(37) \qquad f(n) \text{ is multiplicative if and only if } f'(n) \text{ is.}$$

A function $f(n)$ is called completely multiplicative if the factorization $f(mn) = f(m)f(n)$, where $f(1) = 1$, holds without any restriction of the pair m, n.

* This agrees with the notation of Euler [8], mentioned before (5).

This is the case if and only if the value assigned for $f(n)$ at $n = p^k$ depends only on p. Thus, if $f(n)$ is completely multiplicative, (36) and (4_1) show that

$$(38)\quad f'(1) = 1, \qquad f'(p) = f(p) - 1 \quad \text{and} \quad f'(p^2) = f'(p^3) = \cdots = 0.$$

In other words, if $f'(n)$ is completely multiplicative,(36) and (4_1) imply that $f(p^k)$ is the geometric progression $1 + f'(p) + \cdots + (f'(p))^k$. In particular, if a completely multiplicative function $f'(n)$, or, what is the same thing, the values $f'(p)$ determining it, can attain only the values 1, -1 and 0, it follows that $f(p^k) \geqq 0$ and $f(p^{2k}) \geqq 1$. In view of (37), this entails that $f(n) \geqq 0$ and $f(n^2) \geqq 1$ for every n; so that

$$(39)\qquad \sum f(n)/n^{\frac{1}{2}} = \infty, \qquad \text{where} \quad f(n) \geqq 0.$$

A known variant of Mertens' proof for Dirichlet's theorem on the non-vanishing of the real non-principal L-series at $s = 1$ (cf. pp. 426–435 of Landau's *Handbuch*) is contained in (39) and in what has been italicized before (30). In fact, let $f'(n)$ be a real non-principal character. Then the assumptions of (39) are satisfied. Moreover, since every non-principal character is a periodic function of n and has the mean-value 0 over a period, the partial sums of $\sum f'(n)$ are bounded, and so the series $L(s) = \sum f'(n)/n^s$ converges for $s > 0$. Suppose, if possible, that $L(1) = 0$. Then (i), §4 and the fact italicized before (30) imply that the n-th partial sum of $\sum f(n)$ is $O(n^{\frac{1}{2}})$, which in turn implies that $\sum f(n)/n^s$ converges for $s > \frac{1}{2}$. Consequently, (10) is valid for $s > \frac{1}{2}$, if the function $\zeta(s) = \sum 1/n^s$, where $s > 1$, is thought of as continued analytically for $s < 1$. But $\sum f'(n)/n^s$ converges, and represents therefore a continuous function, for $s > 0$, and so, in particular, at $s = \frac{1}{2}$. Hence, if $s \to \frac{1}{2} + 0$ in (10), there results a contradiction to (39).

17. Let a set, S, be called multiplicative if its characteristic function, $S(n)$, is multiplicative, and let S be called completely multiplicative if the function $S(n)$ is completely multiplicative. On the other hand, let a set of positive integers be called a semi-group if it contains 1 and the product of any two of its elements, including all powers of its elements. Then every semi-group is a multiplicative set and every completely multiplicative set is a semi-group, the situation being as follows:

The most general multiplicative set results if it is decided for every single prime power, p^k, in an arbitrary manner, whether $n = p^k$ (> 1) should or should not be in the set S. In fact, the alternative $S(n) = \frac{1}{2} \pm \frac{1}{2}$, mentioned before (34), is then decided for every $n = p^k$ and therefore, by multiplicativity, for every positive integer n. Clearly, the most general semi-group results if the arbitrary double sequence $S(p^k)$ of the values $\frac{1}{2} \pm \frac{1}{2}$ defining S is restricted by the following condition: If the prime p and the positive integer k are fixed, then either $S(p^{jk}) = 1$ or $S(p^{jk}) = 0$ holds for every positive integer j. Finally, the most general multiplicative set results by choosing an arbitrary set, say R, of primes and placing $S(p) = S(p^2) = \cdots = 1$ or $S(p) = S(p^2) = \cdots = 0$ ac-

cording as the prime p is or is not a prime contained in R. Thus, if r denotes an arbitrary element of the prime set R, a set which may be infinite, finite or vacuous, then a positive integer n is in the completely multiplicative set S if and only if all prime factors of n are primes r. The completely multiplicative set belonging to the prime set R will be denoted by R^*, and R will be called the generator of $S = R^*$.

If S is any multiplicative set, it is a straightforward consequence of the simplest form of the sieve of Eratosthenes (cf. the end of §29 below), that S is measurable in the sense of §15. It follows therefore from the Theorem of §15 as a corollary, that $\sum S'(n)/n$ *is a convergent series for every multiplicative set* S.

Since (6) is the characteristic function, $S(n)$, of the multiplicative set S consisting of the single integer 1, it is seen from (8) that this corollary still contains the prime number theorem. However, while the Theorem of §15 depended, via (vii), on more than the prime number theorem, the above corollary is not deeper than the prime number theorem. In order to show this, it is sufficient to verify that (vi) is applicable in the present case; in fact, (vi) was pointed out to be not deeper than the prime number theorem. But $| S'(p^k) | \leqq 1$, even if S is not multiplicative; cf. the end of §14. It follows therefore from the case $f(n) = S(n)$ of (37) that, if S is multiplicative, then $| S'(n) | \leqq 1$ for every n, and so the assumption, $S'(n) = O_L(1)$, of (vi) is satisfied even in its form $S'(n) = O(1)$.

For an arbitrary multiplicative set, S, let S_0 denote the completely multiplicative set, $S_0 = R^*$, generated by the set, R, of the primes contained in S. Then it is easily seen that $\sum S'(n)/n$ is the Dirichlet product of $\sum S_0'(n)/n$ and of a series which, in view of Euler's factorization, is absolutely convergent. It follows therefore from the multiplication theorem of Mertens-Stieltjes, that the last italicized statement is not weakened if S is replaced by S_0. For this reason, it will amount just to a simplification of the notations, that only completely multiplicative sets will be considered in the sequel.

18. Let r^* run through all integers contained in the completely multiplicative set, $S = R^*$, generated by an arbitrary set, R, of primes, r. Thus, if P denotes the set of all primes p, then $p^* = n = 1, 2, \cdots$. The set, $P - R$, of all primes not contained in a given prime set, R, will be denoted by Q.

Let the Möbius function, $\mu_R(n)$, belonging to a completely multiplicative set, R^*, be defined by placing

$$\mu_R(n) = \mu(n)R^*(n), \tag{40}$$

where $R^*(n)$ denotes the characteristic function of R^* (for instance, the Möbius function $\mu(n)$ is $\mu_P(n)$, since $P^*(n) = 1$ for every n). By the zeta-function belonging to R^* will be meant the function represented in the half-plane $\sigma > 1$, where $s = \sigma + it$, by

$$\zeta_R(s) = \sum (r^*)^{-s}, \quad \text{i.e.,} \quad \zeta_R(s) = \prod (1 - r^{-s})^{-1}. \tag{41}$$

This Euler factorization implies that $\zeta_R(s)\zeta_Q(s)$ is $\zeta_P(s)$, that is, Riemann's $\zeta(s)$. Hence, (10) shows that $\sum f(n)/n^s = \zeta_Q(s)$ when $\sum f'(n)/n^s = 1/\zeta_R(s)$. On the other hand, from (40),

$$\sum \mu_R(n)/n^s = \sum \mu(r^*)/(r^*)^s = \prod(1 + \mu(r)/r^s),$$

and so, since $\mu(p) = -1$ for every prime p,

$$(42)\qquad \sum \mu_R(n)/n^s = \prod(1 - r^{-s}); \quad \text{so that} \quad 1/\zeta_R(s) = \sum \mu_R(n)/n^s,$$

by the second of the relations (41). But the first of the relations (41), when applied to Q, q^* instead of R, r^*, can be written in the form $\zeta_Q(s) = \sum Q^*(n)/n^s$, where $Q^*(n)$ denotes the characteristic function of the completely multiplicative set Q^*. Consequently, the second of the relations (42) and the conclusion obtained from (41) and (10) imply that

$$(43)\qquad f(n) = Q^*(n) \quad \text{when} \quad f'(n) = \mu_R(n), \quad \text{where} \quad Q = P - R.$$

Hence, if the italicized result of §17 is applied to $S = Q^*$, it follows that $\sum \mu_R(n)/n$ is convergent.

In view of (42), all of this can be summarized by saying that, if R is any set of primes, and *if* $\mu_R(n)$ *is defined by* (41) *and* (42), *then* $\sum \mu_R(n)/n$, *i.e., the Dirichlet series of the function* $1/\zeta_R(s)$ *at the point* $s = 1$, *is a convergent series.* The prime number theorem follows by choosing $R = P$; cf. the end of §8. However, in terms of the axiomatic vernacular of the theory of primes with regard to comparative "depths," the convergence of $\sum \mu_R(n)/n$ for an arbitrary R is not deeper than the prime number theorem, since (vi) was applicable in §17.

19. This does not imply that it is easy, or at least possible, to infer the convergence of an arbitrary $\sum \mu_R(n)/n$ from the convergence of $\sum \mu(n)/n$ (although the non-vanishing of $\zeta(1 + it)$ for $-\infty < t < \infty$ is a trivial conclusion from the convergence of $1/\zeta(1) = \sum \mu(n)/n$). In fact, it will now be shown that the usual terminology with regard to comparative depths is quite inadequate in describing the situation, unless one specifies *which* general Tauberian theorem is to be combined with the non-vanishing of Riemann's ζ on the line $s = 1 + it$.

The proof (§23) for the necessity of such a specification will depend on the construction of an R for which $\zeta_R(s)$ becomes sufficiently pathological on the line $\sigma = 1$ (§22). In order to carry out this construction, a fact relating to Riemann's $\zeta(s)$ will first be established (§20).

The simplest (and for the above purpose insufficient) result will be (§21) that *there exists for every given* $t \neq 0$ *a set,* $R = R_t$, *of primes for which the series* $1/\zeta_R(s) = \sum \mu_R(n)/n^s$ *becomes divergent at the point* $s = 1 + it$. Thus the italicized result of §18, where $t = 0$, is exceptional.

Another interpretation of this negative result is that the possibility of generalizing the classical case of P to the case of any R in the result of §18 is a coincidence. In fact, if $P = R$, it is known that $1/\zeta(s) = \sum \mu(n)/n^s$ converges at every point of the line $s = 1 + it$. Actually, the convergence is uniform on

every bounded t-segment *because* the series $\sum \mu(n)/n$, belonging to $t = 0$, is convergent. The truth of this implication is supplied by the Tauberian theorem of P. Fatou-M. Riesz.

20. Let $\omega(s)$ denote the function represented in the half-plane $\sigma > 1$ by the Dirichlet series $\sum p^{-s}$, where p runs through all primes. The latter should be thought of arranged in increasing order, if the series is not absolutely convergent. According to Euler's factorization of $\zeta = \zeta_P$, the function $\omega(s)$ is the essential part of $\log \zeta(s)$ in the half-plane $\sigma > \frac{1}{2}$ (cf. the case $R = P$ of (46) below). However, as shown by Kluyver under Riemann's hypothesis, and then by Landau and Walfisz without this hypothesis, the derivative $d\omega(s)/ds$ is meromorphic in the half-plane $\sigma > 0$ but has the line $\sigma = 0$ as natural boundary; in this connection, cf. Estermann [7].

A result of Mertens [30], when combined with the prime number theorem, states that the series obtained by placing $s = 1 + it$ in $\omega(s) = \sum p^{-s}$ is uniformly convergent on every closed bounded t-interval not containing $t = 0$ (today, this result may be inferred by combining the Tauberian theorem quoted at the end of §19 with the regularity of the function $\omega(s)$ at every point $t \neq 0$ of the line $s = 1 + it$). In particular, the real part of the series $\sum p^{-1-it}$, i.e.

$$\sum p^{-1} \cos (t \log p), \tag{44}$$

is uniformly convergent for $\epsilon < |t| < 1/\epsilon$, if $\epsilon > 0$. However, what will be needed in §21 is not only this positive result but a negation, to the effect that (44) is not absolutely convergent for some $t \neq 0$. It is not more difficult to show that *for no t is* (44) *absolutely convergent.*

In order to prove this negation, use will be made of the following fact: If λ is a positive number and if J denotes an interval, then there exists a positive $c = c(\lambda, J)$ such that more than cx of the first $[x]$ prime number logarithms, $\log p$, where $p = p_n$ and $p_n < p_{n+1}$, are mod λ within J, as $x \to \infty$ (for a sharper result, cf. Wintner [43]). This obviously implies that there exists for every real number t a positive number $c = c(t)$ such that, if $\epsilon_n = \epsilon_n(t)$ denotes 1 or 0 according as $|\cos (t \log p_n)|$ does or does not exceed $\frac{1}{2}$, then $\phi(x) > cx$ as $x \to \infty$, where $\phi(x) = \epsilon_1 + \cdots + \epsilon_n$ if $n \leqq x < n + 1$. On the other hand, since $p_n < 2n \log n$ for every large n, it is clear from the definition of ϵ_n that the sum of the absolute values of the first $[x]$ terms of the series (44) cannot be less than

$$\frac{1}{2} \sum_{n=1}^{x} p_n^{-1} \epsilon_n > \frac{1}{2} \sum_{n=a}^{x} (2n \log n)^{-1} \epsilon_n = 2^{-2} \int_a^x (u \log u)^{-1} \, d\phi(u),$$

where a is a constant. Since $\phi(x)$ is the $[x]$-th partial sum of the series $\sum \epsilon_n$, where $0 \leqq \epsilon_n \leqq 1$, it is also clear that $\phi(x) = O(x)$, and therefore $(x \log x)^{-1}\phi(x) = o(1)$, as $x \to \infty$. Hence, a partial integration shows that the integral in the last formula line is asymptotically proportional to

$$-\int_a^x \phi(u) \, d(u \log u)^{-1} = \int_a^x \phi(u)(u \log u)^{-2}(1 + \log u) \, du.$$

If $\phi(u)$ is estimated by the above inequality, $\phi(u) > cu$, valid for every large u and for a certain positive c which is independent of u, it is seen that the last integral exceeds a fixed multiple of $\log \log x$, as $x \to \infty$. Consequently, the sum of the absolute values of the first $[x]$ terms of the series (44) exceeds a fixed multiple of $\log \log x$, if t is fixed and $x \to \infty$. This proves the assertion italicized after (44).

21. It is now easy to see that Mertens' conclusion from the prime number theorem, mentioned before (44), becomes false if the set, P, of all primes p is replaced by a suitable set, R, of primes.

First, if R is any set of primes r, let $\omega_R(s)$ denote the function represented by the Dirichlet series

$$\omega_R(s) = \sum_R r^{-s} \tag{45}$$

($\sigma > 1$). Then (41) shows that $\log \zeta_R(s) = \sum_{k=1}^{\infty} \omega_R(ks)/k$, i.e.,

$$\log \zeta_R(s) - \omega_R(s) = \sum_{k=2}^{\infty} \sum_r k^{-1} r^{-ks}, \qquad \text{if} \quad \sigma > \tfrac{1}{2}. \tag{46}$$

In fact, the absolute convergence of (45) for $\sigma > 1$ implies the absolute convergence of the Dirichlet series (46) for $\sigma > \frac{1}{2}$. In particular, the deviation, (46), of $\log \zeta_R(s)$ from $\omega_R(s)$ is a function regular in the half-plane $\sigma > \frac{1}{2}$ (however, it will be seen in §22 that $\omega_R(s)$, hence $\log \zeta_R(s)$, can have the line $\sigma = 1$ as natural boundary).

For a fixed $t \neq 0$, let $R = R^t$ denote either the set of those primes, r, for which $\cos(t \log r)$ is positive or of those primes, r, for which $\cos(t \log r)$ is negative. Since the series (44) is convergent without being absolutely convergent (§20), the series

$$\sum_R r^{-1} \cos(t \log r) \tag{47}$$

then diverges to $\pm\infty$. But (45) shows that (47) is the real part of the series of $\omega_R(1 + it)$. On the other hand, since the Dirichlet series (46) is convergent for $\sigma > \frac{1}{2}$, the difference on the left of (46) is continuous, and has therefore a continuous real part, for $\sigma \geqq 1$. Thus it is clear from the Abelian continuity theorem of Dirichlet series, that the real part of $\log \zeta_R(1 + \epsilon + it)$ tends to $\pm\infty$ if $0 < \epsilon \to 0$, where t ($\neq 0$) is the fixed superscript of the prime set $R = R^t$. Since this means that $\log |\zeta_R(1 + \epsilon + it)| \to \pm\infty$, it follows that the (complex-valued) function $\zeta_R(s) = \zeta_R(1 + \epsilon + it)$ tends to infinity or to zero according as the upper or the lower sign is chosen.

If the lower sign is chosen, it is seen from the two identities (42), that an application of the Abelian continuity theorem of Dirichlet series completes the proof of the italicized assertion of §19.

22. The above construction may be refined as follows:

Let $t_1, t_2, \cdots$, be a real sequence consisting of non-vanishing rational numbers and containing every non-vanishing rational number an infinity of times. Starting with t_1, choose a prime set R_1 for which the series (47) belonging to $R = R_1$, $t = t_1$ diverges to $-\infty$, and let m_1 be an integer for which the m_1-th partial sum of this numerical series is less than -1. If $R_1, \cdots, R_{n-1}$ and $m_1, \cdots, m_{n-1}$ have been defined for a fixed n, choose a prime set R_n so as to satisfy the following conditions: R_n contains the first m_{n-1} primes occurring in R_{n-1}, and the series (47) belonging to $R = R_n$, $t = t_n$ diverges to $-\infty$. Finally, choose an integer m_n so that the m_n-th partial sum of the numerical series just mentioned is less than $-k_n$, where $k_1 = 1, k_2, \cdots$ is an *increasing* sequence of integers, to be subject to certain restrictions the nature of which will become clear in a moment.

Now let R be defined as the set of those primes, r, which occur among the first m_n primes contained in R_n for some, hence for every, sufficiently large value of n. Then, since every non-vanishing rational number occurs in $t_1, t_2, \cdots$ an infinity of times, it is clear that the lower limit of the partial sums of the series (47) belonging to any non-vanishing rational t is $-\infty$, if the increasing sequence $k_1 = 1, k_2, \cdots$ mentioned before is suitably chosen in the definition of R. It follows therefore from (45) and (46) in exactly the same manner as in §21, that the present R is such as to satisfy the condition

$$\liminf_{\epsilon \to 0} \log |\zeta_R(1 + \epsilon + it)| = -\infty \tag{48}$$

for every non-vanishing rational t. The Abelian continuity theorem, used in §21, must of course be replaced by the corresponding Abelian inequality, estimating the lower and upper boundary limits of the generating function in terms of the lower and upper limits of the partial sums belonging to the boundary point.

Since (48) holds for a set of the t-values which is dense on the t-axis, it is clear that *the present* $\zeta_R(s)$ *has the line* $\sigma = 1$ *as natural boundary*; in fact, the function (41), being regular and distinct from zero in the half-plane $\sigma > 1$, cannot vanish identically. Incidentally, since (48) holds for a dense set of t-values, it follows from a well-known property of sequences of continuous functions, that (48) must hold for every t contained in a set which is of the second category, and therefore non-enumerable, on every t-interval.

Only the lower sign of the alternative, $\pm\infty$, following (47) was used in the above construction. If the upper sign is used, it follows that all of the above remarks remain valid if (48) is replaced by

$$\limsup_{\epsilon \to 0} \log |\zeta_R(1 + \epsilon + it)| = +\infty. \tag{49}$$

It is also clear that the construction can be modified so as to lead to a prime set R satisfying (48) on a dense t-set and (49) on another dense t-set.

23. Since Riemann's $\zeta(s) = \sum 1/n^s$ has a simple pole at $s = 1$, it is clear that the Ikehara's Tauberian theorem (cf. Wiener [40], p. 127) may, after subtraction of $\zeta(s)$, be restated as follows: If a Dirichlet series $G(s) = \sum g(n)/n^s$ converges for $\sigma > 1$, and if the function $G(s)$ goes over into a continuous boundary function $G(1 + it)$ on the line $\sigma = 1$, then $g(1) + \cdots + g(n) = o(n)$ holds whenever $g(n) = O_L(1)$.

In particular, the prime number theorem follows in the form (12) by choosing $G(s)$ to be $1/\zeta(s) = \sum \mu(n)/n^s$, and using the regularity of the function $1/\zeta(1 + it)$ for $-\infty < t < \infty$, that is, the non-vanishing of $\zeta(1 + it)$. But if $\zeta = \zeta_P$ and $\mu = \mu_P$ are replaced by ζ_R and μ_R respectively, where R is an arbitrary prime set, it is seen from (42) and from the possibility of (48) on a dense t-set, that Ikehara's Tauberian theorem is incapable of supplying the estimate which results if μ is generalized to μ_R in (12); an estimate which, in view of (24), is implied by the italicized result of §18. On the other hand, Wiener's particular case of the general theorem of Hardy and Littlewood (cf. §6 above) is a Tauberian theorem, which, when combined with the non-vanishing of $\zeta(1 + it)$, supplies not only the prime number theorem but (vi) as well, and so it suffices for the proof of the italicized result of §18. Accordingly, §22 proves that the Lambertian approach, based on the behavior of a generating function on a *real interval*, is methodically superior to the approach depending on the behavior of Dirichlet's generator in a *complex half-plane.*

It is true that the proof of Ikehara's theorem remains valid if the existence of a continuous boundary function $G(1 + it)$ is replaced by the requirement that the inequality $| G(\sigma + it) | < F(t)$, where $1 < \sigma < 2$ and $-T \leqq t \leqq T$, should hold for a function $F(t)$ which is L-integrable on the interval $-T \leqq t \leqq T$, if T is arbitrarily fixed. However, the constructions in §22 make hard indeed the satisfaction of any such condition in the case $G(s) = 1/\zeta_R(s)$ of §18, if R is suitably chosen.

24. It cannot, of course, be expected that a prime set R constructed in the fashion of §22 or §21 will have the characteristics of an "average" prime set R, selected from the set P of all primes "at random." The case of such a "generic" R will now be considered. This notion becomes a mathematical one, if an ordinary Lebesgue measure is defined on the space of all possible selections, as follows:

Let $(\pm)$ be a space consisting of two points, $+$ and $-$, and carrying that measure function for which the measure of either point is $\frac{1}{2}$; so that the space $(\pm)$ is of measure 1. Consider the infinite product space $(\pm) \times (\pm) \times \cdots$, consisting of points $(\pm, \pm, \cdots)$ each of which represents an arbitrary decision of each of the individual alternatives $\pm, \pm, \cdots$, and define the product measure of the measures carried by the individual factor spaces $(\pm)$, $(\pm)$, $\cdots$ to be the measure carried by the product space; so that the latter is again of measure 1. The customary realizations of a point, $+$ or $-$, in an individual factor space, of a point in the product space and of the product measure on the latter are the pair, 0 and 1, of "dyadic digits", a binary expansion of any non-negative number

not exceeding unity and the Euclidean Lebesgue measure on the interval $0 \leqq \xi \leqq 1$ respectively (the numbers ξ possessing two binary expansions, being enumerable, are of zero measure and can therefore be neglected). In the sequel, such expressions as "space," "set," "point," "measure," "almost all," "choice" will always refer to the product space and never to its factors.

If $f(1), f(2), \cdots$ is a sequence, an arbitrary (infinite or finite, possibly vacuous) subsequence of it results by choosing an arbitrary point $(\pm, \pm, \cdots)$ and then omitting those terms of the corresponding sequence $(\frac{1}{2} \pm \frac{1}{2})f(1)$, $(\frac{1}{2} \pm \frac{1}{2})f(2)$, $\cdots$ for which $\frac{1}{2} \pm \frac{1}{2}$ is $\frac{1}{2} - \frac{1}{2} = 0$. This introduces a Lebesgue measure on the space of all order-preserving selections from $f(1), f(2), \cdots$ in such a way that the measure of the space of all selections is 1. Finally, corresponding to the $(\pm, \pm, \cdots)$-representation of the subsequences of a sequence $f(1), f(2), \cdots$, an arbitrary subseries of a series $\sum f(n)$ can be defined to be $\sum (\frac{1}{2} \pm \frac{1}{2})f(n)$.

A general theorem, due to Khintchine and Kolmogoroff, for which today various proofs are known (the simplest seems to be that given by Lévy [26]), can now readily be applied. It leads, after obvious reductions, to the result that *almost all or almost none of the subseries of a given series* $\sum f(n)$ *are divergent according as either or neither of the series* $\sum f(n)$, $\sum |f(n)|^2$ *is divergent.*

In the sequel, it will be sufficient to use this fact in that substantially weakened form which *assumes* the convergence of $\sum f(n)$ and makes therefore only the assertion that almost all subseries of a *convergent* series $\sum f(n)$ are convergent or divergent according as $\sum |f(n)|^2 < \infty$ or $\sum |f(n)|^2 = \infty$. Actually, this corollary of the true theorem is equivalent to that particular case of the criterion of Khintchine and Kolmogoroff according to which the alternative $\sum |f(n)|^2 \lesseqgtr \infty$ decides whether almost all of the series $\sum \pm f(n)$ are convergent or divergent.

25. If $f(n)$ is replaced by $f(n)/n^{\frac{1}{2}+\epsilon}$, where $\epsilon > 0$ is fixed, it follows from this corollary that almost all series $\sum \pm f(n)/n^{\frac{1}{2}+\epsilon}$ are convergent if $f(n) = O(1)$ and so, in particular, if $f(n) = 1$ for every n. Since the sum of a sequence of zero sets is a zero set, this implies that almost all Dirichlet series $\sum \pm 1/n^s$ are convergent in the half-plane $\sigma > \frac{1}{2}$. But, if $\lambda(n)$ is Liouville's coefficient, that is, the completely multiplicative function which becomes -1 for every prime, then, on the one hand, $\sum \lambda(n)/n^s$ is of the form $\sum \pm 1/n^s$ and, on the other hand, $\sum \lambda(n)/n^s = \zeta(2s)/\zeta(s)$ for $\sigma > 1$. Since the convergence of the latter series for $\sigma > \frac{1}{2}$ would obviously imply the non-vanishing of $\zeta(s)$ in the half-plane $\sigma > \frac{1}{2}$, the convergence of almost all series $\sum \pm 1/n^s$ in the half-plane $\sigma > \frac{1}{2}$ has repeatedly been interpreted to the effect that Riemann's hypothesis is true at any rate for almost all zeta-functions (Wiener, Jessen, Paley and Wiener, Cramér).

However, such an interpretation is hardly legitimate, since it neglects the arithmetical nature of the problem and, in particular, the obvious fact that almost all of the functions $g(n)$ defined by $\sum g(n)/n^s = \sum \pm 1/n^s$ are not multiplicative; an objection first pointed out (in a slightly different form) by Lévy [26], p. 155.

Actually, it will turn out in §26 that, if the arithmetical nature of the problem is not neglected, *Riemann's hypothesis, instead of being almost always true, is almost always true or almost always false according as it is true or false for* $\zeta(s) = \sum 1/n^s$ *itself.*

In order to exclude zeta-functions for which the condition of multiplicativity is violated, the product space of the selections must be based on the sequence P of all primes, p, and not on arbitrary integers. To this end, let an arbitrary set R of primes r be thought of as introduced as follows: p is or is not an r according as $(\frac{1}{2} \pm \frac{1}{2})p = p$ or $(\frac{1}{2} \pm \frac{1}{2})p = 0$. This identifies the space of all the prime sets R with the $(\pm, \pm, \cdots)$-space considered in §25. Correspondingly, an R (and any such functional of R as $\zeta_R(s)$, $\mu_R(n)$, R^*, $\cdots$, each of which determines R) will be said to belong to *a random selection*, or to be *a random* R, if that point of the $(\pm, \pm, \cdots)$-space which represents R is not contained in a set of measure 0. The excluded 0-set, being unspecified, will not be always the same set.

26. Since the series $\sum (p^{-\frac{1}{2}-\epsilon})^2$ converges for every $\epsilon > 0$ but not for $\epsilon = 0$, the theorem mentioned at the end of §24 implies that the abscissa of convergence of the Dirichlet series $\sum \pm p^{-s}$, where p runs through all primes, is $\frac{1}{2}$ for almost all choices of $(\pm, \pm, \cdots)$. Incidentally, it follows from a general theorem (cf. Carlson [4]) on Dirichlet series, that the line $\sigma = \frac{1}{2}$ is a natural boundary of almost all functions $\sum \pm p^{-s}$.

If R is any prime set, $\sum r^{-s} = \sum (\frac{1}{2} \pm \frac{1}{2})p^{-s}$ holds in the notations of §25. According to (45), this can be written in the form $\omega_R(s) = \frac{1}{2}\omega(s) + \frac{1}{2}\sum \pm p^{-s}$, where $\omega(s) = \omega_P(s)$. Furthermore, (46) and the case $R = P$ of (46) show that $\omega_R(s)$ and $\omega(s)$ respectively differ from $\log_R(s)$ and $\log \zeta(s)$ only in additive terms which possess convergent Dirichlet series for $\sigma > \frac{1}{2}$. Hence, it is clear from the convergence of almost all series $\sum \pm p^{-s}$ in the half-plane $\sigma > \frac{1}{2}$ that, unless R belongs to a set of measure 0, the difference of $\log \zeta_R(s)$ and $\frac{1}{2} \log \zeta(s)$ is a Dirichlet series which converges for $\sigma > \frac{1}{2}$ and represents a function possessing the line $\sigma = \frac{1}{2}$ as natural boundary. Since any Dirichlet series represents a regular function within its domain of convergence, it follows that, *for a random* R, *the function* $\zeta_R(s)/\zeta(s)^{\frac{1}{2}}$ *is regular and distinct from* 0 *in the half-plane* $\sigma > \frac{1}{2}$.

This implies the correctness of what has been said at the beginning of §24 and proves, in addition, the italicized statement of §25. It also follows that, if Riemann's hypothesis be false, the numerical values of those zeros of $\zeta(s)$ which are within the half-plane $\sigma > \frac{1}{2}$ ought to have a "universal", rather than an *arithmetical*, significance, since they must be zeros of "almost every" $\zeta_R(s)$ and vice versa.

Since $\zeta(s)$ is regular and non-vanishing at every point $s \neq 1$ of the line $\sigma = 1$, the last italicized statement entails (without any hypothesis) that every random $\zeta_R(s)$ is regular and non-vanishing at every point $s \neq 1$ of the line $\sigma = 1$. Consequently, the Tauberian theorem referred to at the end of §19, when combined with the italicized result of §18, implies that, *for a random* R, *the series* $1/\zeta_R(s) = \sum \mu_R(n)/n^s$ *converges at every point of the line* $\sigma = 1$; *however, its abscissa of convergence is* $\sigma = 1$.

The truth of the last assertion follows from the fact that, as shown above, a random $\zeta_R(s)$ behaves at $s = 1$ in the same way as the function $\zeta(s)^{\frac{1}{2}} \sim (s - 1)^{-\frac{1}{2}}$; so that the function $1/\zeta_R(s)$ cannot be regular at $s = 1$, and therefore its Dirichlet series, $\sum \mu_R(n)/n^s$, cannot converge beyond the line $\sigma = 1$. (This fact, which does not depend on Riemann's hypothesis, is contrasted by Littlewood's result, according to which Riemann's hypothesis is true if and only if the abscissa of convergence of the Dirichlet series, $\sum \mu(n)/n^s$, of $1/\zeta(s)$ itself is $\frac{1}{2}$).*

27. The result of §26 on the analytic behavior of a random $\zeta_R(s)$ for $\sigma > \frac{1}{2}$ will now be completed by proving that, *for a random R, the square of $\zeta_R(s)$ is a function meromorphic in the half-plane $\sigma > \frac{1}{2}$ and possessing the line $\sigma = \frac{1}{2}$ as natural boundary.* The first of these two statements is implied by §26. The second will be proved by showing that, for a random R, the line $\sigma = \frac{1}{2}$ is natural boundary of $\log \zeta_R(s)$.

First, if $\sigma > 1$, then, whether R is or is not random, a Möbius inversion proves the equivalence of the two relations

$$(50_1)\qquad \log \zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n}\, \omega_R(ns); \qquad\qquad (50_2)\qquad \omega_R(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta_R(ns),$$

the first of which was pointed out after (45). On the other hand, it is clear from §26 that, for a random R, the logarithm of $\zeta_R(s)$ is regular at every point $s \neq 1$ of the line $\sigma = 1$ (since $\zeta(s) \neq 0$ on this line). Now let the first term of the series (50_2) be separated from all the terms belonging to an $n > 1$. It then follows that, if the logarithm of a random $\zeta_R(s)$ were regular at a point $s \neq \frac{1}{2}$ of the line $\sigma = \frac{1}{2}$, the relation (50_2), where $\sigma > 1$, would imply (by uniform convergence) that $\omega_R(s)$ is regular at the same s and for the same R. But this is impossible, since, as mentioned at the beginning of §26, the function $\omega_R(s)$ belonging to a random R is singular at every point of the line $\sigma = \frac{1}{2}$. This contradiction completes the proof.

It is clear from §25 that, in view of that formulation of the law of large numbers which is due to Borel [3], almost all selections of a prime set R have the property that, as $x \to \infty$, the number of primes contained in R and not exceeding x is asymptotically equal to half the number of all primes not exceeding x. This necessary condition for a random R is known to be satisfied if R consists of all primes of the form $4m + 1$. However, $\zeta_R(s)$ does not have the natural boundary $\sigma = \frac{1}{2}$ in this case.

In fact, if r and q respectively run through the sets, R and Q, of the primes of the form $4m + 1$ and $4m + 3$, then the series $\sum (-1)^{n+1}/(2n - 1)^s$, where $\sigma > 1$ (cf. (11), §2), is identical with $\prod(1 - r^{-s})^{-1}\prod (1 + q^{-s})^{-1}$. But the

* The situation becomes quite different if (42) is replaced by the Dirichlet series defined by

$$(*)\qquad \sum \beta_R(n)/n^s = \prod (1 \pm p^{-s}),$$

($\sigma > 1$), where p runs through *all* primes and the lower or the upper alternative sign is chosen according as p is or is not in R (cf. §41-§46 below).

series represents the non-principal L-function (mod 4), while the two products are $\zeta_R(s)$ and $\zeta_Q(2s)/\zeta_Q(s)$, by (41). On the other hand, since the set of all primes consists of R, Q and the even prime, $\zeta(s)$ is the product of $\zeta_R(s)$, $\zeta_Q(s)$ and $(1 - 2^{-s})^{-1}$. If these representations of $L(s)$ and $\zeta(s)$ are multiplied and if s is replaced by $2s$ in the representation of $\zeta(s)$, there result two relations which, when divided, supply the identity

$$\zeta_R(s)^2/\zeta_R(2s) = (1 + 2^{-s})^{-1}L(s)\zeta(s)/\zeta(2s). \tag{51}$$

Since the function on the right of (51) is meromorphic in the whole plane, it follows (by first considering the half-plane $\sigma > 1$, then replacing s by $\frac{1}{2}s$, the resulting s again by $\frac{1}{2}s$ and so on), that the numerator on the left of (51), instead of having the natural boundary $\sigma = \frac{1}{2}$, is meromorphic in the half-plane $\sigma > 0$.

It is also seen from (51) that $L(s)\zeta(s)/\zeta_R(s)^2$ is a non-vanishing regular function in the half-plane $\sigma > \frac{1}{2}$. Hence, if the present R were random in the sense of the italicized statement of §25, it would follow that Riemann's hypothesis for $\zeta(s)$ implies the analogue of Riemann's hypothesis for $L(s)$, and therefore, according to Hardy and Littlewood [15], the truth of Chebyshev's assertion concerning the primes of the respective forms $4m \pm 1$.

28. For the sake of completeness, consider now

$$\sum p^{-1} \sin (t \log p), \tag{52}$$

the series conjugate to (44). Clearly, the proof given in §20 for (44) also shows that (52) is not absolutely convergent, if $t \neq 0$. Furthermore, the fact mentioned before (44) implies that (52) is uniformly convergent for $\epsilon < |t| < 1/\epsilon$, $\epsilon > 0$. But, in contrast with (44), the series (52) converges at $t = 0$ also. It will now be shown that *the series* (52) *possesses near* $t = 0$ *a Gibbs phenomenon, in the sense that its partial sums are uniformly bounded but not uniformly convergent near* $t = 0$; in fact, as $x \to \infty$, the difference

$$\sum_{p \leqq x} p^{-1} \sin (t \log p) - \mathrm{Si}\,(t \log x), \quad \text{where} \quad \mathrm{Si}\, u = \int_0^u \frac{\sin v}{v}\, dv, \tag{53}$$

tends to a limit function uniformly for $|t| <$ const.

First, if $\pi(x)$ denotes the number of primes not exceeding x and $\mathrm{Li}(x)$ the integral logarithm, then $\pi(x) \sim \mathrm{Li}(x) \sim x/\log x$, and the remainder term of $\pi(x) - x/\log x$ is $O(x/\log^2 x)$. This estimate is of about the same depth as (13); it is contained in de la Vallée-Poussin's result, according to which the asymptotic series of $\mathrm{Li}(x)$ is asymptotic to $\pi(x)$ also. But $\pi(u) = u/\log u + O(u/\log^2 u)$ as $u \to \infty$ implies that, as $x \to \infty$,

$$\sum_{p \leqq x} p^{-1} \sin (t \log p) = \int_2^x u^{-1} \sin (t \log u)\{d(u/\log u) + dO(u/\log^2 u)\}. \tag{53*}$$

A partial integration shows that the contribution of $dO(u/\log^2 u)$ to (53*) is of the form

$$O(1/\log^2 x) + O(1) - \int_2^x O(u/\log^2 u)\, d_u(u^{-1} \sin t(\log u)),$$

where, as always in the sequel, the O-functions are uniform for $-\infty < t < \infty$. Since the last integrand is

$$O(u/\log^2 u) \mid tu^{-1}O(1)u - O(1) \mid /u^2 = (1 + |t|)\, O(u \log^2 u)^{-1},$$

and since $(u \log^2 u)^{-1}$ has an absolutely convergent integral over the range $2 \leqq u < \infty$, it follows that, if the contribution of $dO(u/\log^2 u)$ to (53*) is omitted, the error resulting in (53*) tends to a limit function uniformly for $|t| <$ const., as $x \to \infty$. On the other hand, if $d(u/\log u)$ is replaced by $(\log u)^{-1}\, du$, then the error committed, being identical with $-(\log u)^{-2}du$, contributes to (53*) a term which has the absolute value

$$\left| \int_2^x -(\log u)^{-2} u^{-1} \sin (t \log u)\, du \right| < \int_2^x (u \log^2 u)^{-1}\, du,$$

and tends therefore to a limit function uniformly for all t, as $x \to \infty$. Hence, in order to complete the proof of the statement made with regard to the difference (53), it is sufficient to observe that, if $\{\quad\}$ in (53*) is replaced by $(\log u)^{-1}du$, then the integral on the right of (53*) goes over into

$$\int_2^x (u \log u)^{-1} \sin (t \log u)\, du = \int_{t \log 2}^{t \log x} v^{-1} \sin v\, dv, \qquad (v = t \log u).$$

CHAPTER III

THE GENERATORS

29. For any prime set, R, and for the completely multiplicative set, R^*, generated by R, let $\pi_R(s)$ and $[x]_R$ denote the number of those integers, r and r^*, which do not exceed x and are contained in R and R^* respectively; so that, in terms of the characteristic functions, $R(n)$ and $R^*(n)$,

$$(53_1) \qquad \pi_R(x) = \sum_{n=1}^{x} R(n); \qquad (53_2) \quad [x]_R = \sum_{n=1}^{x} R^*(n).$$

In particular, $\pi_P(x)$ is the number, $\pi(x)$, of all primes not exceeding x, while $[x]_P$ is identical with the symbol, $[x]$, of the greatest integer.

It can be expected that, corresponding to Euler's dictum according to which $\sum p^{-1}$ is the logarithm of the harmonic series $\sum n^{-1}$ (cf. Euler [9]), there exists a certain "logarithmic" connection between the asymptotic orders of $\pi_R(x)$ and $[x]_R$ for an arbitrary R, as $x \to \infty$; a connection which, if R is thought of as the "infinitesimal generator" of the arithmetical semi-group R^*, corresponds to Lie's exponential representation of a continuous cyclic semi-group. The present chapter deals with an analysis of this connection.

The most primitive result in this direction may be formulated as follows: If Q denotes the prime set complementary to a prime set R, so that

$$(54) \quad \zeta_R(s)\zeta_Q(s) = \zeta(s), \text{ where } \zeta_R(s) = \prod (1 - r^{-s})^{-1}, \quad \zeta_Q(s) = \prod (1 - q^{-s})^{-1},$$

$(\sigma > 1)$, then *the asymptotic relation*

$$(55) \qquad [x]_R \sim x/\zeta_Q(1) \quad \text{as} \quad x \to \infty$$

holds for every R, with the understanding that

$$(55 \text{ bis}) \qquad [x]_R = o(x) \quad \text{if} \quad 1/\zeta_Q(1) = 0, \quad \text{i.e.,} \qquad \text{if} \quad \sum q^{-1} = \infty.$$

The equivalence of the two assumptions of (55 bis) is clear from the representation of $\zeta_Q(s)$ in (54), since the product $\prod (1 - q^{-1})$ is 0 when $\sum q^{-1} = \infty$.

If the prime set R is such as to make the series $\sum q^{-1}$ convergent, then, since $1/\zeta_Q(1) = \prod (1 - q^{-1})$, the relation (55) is clear from the simplest form of the sieve of Eratosthenes. But if (55) is true for the case $1/\zeta_Q(1) > 0$, then (55 bis) follows, for reasons of monotony, by first applying (55) to the case in which Q consists of a finite number, N, of primes q, and then letting $N \to \infty$.

According to (53_2) and the beginning of §15, the content of (55), (55 bis) can be expressed by saying that every R^* is measurable and has a positive or a vanishing measure according as the sum of the reciprocal values of all primes not contained in the set is convergent or divergent. Actually, since the reciprocal values of all but the first powers of all the primes form a convergent double series, it is clear from the proof that the measurability of the set and the criterion

for the vanishing of the measure remain unaltered if the set R^*, which in the terminology of §17 is completely multiplicative, is replaced by an arbitrary multiplicative set.

30. It may be mentioned that this elementary observation supplies the proof for a general assertion of Ramanujan, published from his manuscripts by Watson [39]. By using the machinery of the prime number theorem, Watson ([39]; cf. Hardy [14], pp. 167–169) succeeded in proving Ramanujan's assertion for the particular case in which the underlying prime set (Q in the notation used below) corresponds to an arithmetical progression; a particular case which, as shown by Watson, suffices in the proof of Ramanujan's asymptotic congruency properties of $\sigma_s(n)$ and of his τ-function.

The machinery of the prime number theorem, when applicable, supplies, of course, more than what is needed. However, this machinery, which fails in the general case, can be disposed of, and so it is hard to believe that Ramanujan did not have a proof for his assertions.

If n is a positive integer and p is a prime, let n_p denote the greatest non-negative integer k for which the k-th power of p divides n. If Q is any set of primes, let S_Q denote the set of those positive integers n which have the property that n_p is even for every p contained in Q. Then Ramanujan's general assertion is that S_Q is of measure 0 if (and only if) the sum of the reciprocal values of all primes contained in Q is divergent.

In view of the end of §29, the truth of this assertion follows if it is ascertained that (a) the set S_Q is multiplicative for every Q and (b) a prime is not contained in S_Q if and only if it is contained in Q. But (a) follows from the fact that the sum of two even exponents is even (and that 0 is an even exponent). And (b) follows from (a) by observing that, if n is a prime, then n_p is 0 (hence even) or 1 (hence odd) according as the prime n is not or is exactly that p which is the subscript of n_p.

31. The general treatment of the "logarithmic" problem announced at the beginning of §29 will be based on an elementary lemma which, when formulated only for the relevant case of ordinary Dirichlet series, may be stated as follows: *If a real Dirichlet series* $\sum a(n)/n^s$ *has non-negative coefficients and converges for every* $s > 0$, *then*

$$\sum_{n=1}^{\exp(1/s)} a(n) \sim \sum_{n=1}^{\infty} a(n)/n^s \quad \text{as} \quad s \to 0; \qquad (s > 0). \tag{56}$$

This is trivial if $\sum a(n)/n^s$ converges at $s = 0$.

The proof of this lemma requires only a combination of known facts scattered in the literature.

In order to see this, let $L(x)$ be a real-valued function defined for large positive values of x and satisfying $L(x) \to \infty$ as $x \to \infty$. Such a function is said to be *slow* if $L(\theta x) \sim L(x)$ as $x \to \infty$ holds for every positive constant θ. It is easy to

see that, if $L(x)$ is assumed to be *monotone*, the condition of slowness implies the estimate $L(x) = O(x^\epsilon)$, where $\epsilon > 0$ is arbitrarily small and $x \to \infty$. It seems to have been forgotten that precisely this implication is the historical origin of the notion of a "slow" function. In fact, if $L(x)$ is any *monotone* function satisfying $L(x) \to \infty$, and if $L(x) = O(x^\epsilon)$ holds for every fixed $\epsilon > 0$, then $L(\theta x) \sim L(x)$ must hold for every fixed $\theta > 0$. This observation is due to Pringsheim [33], who therefore appears to be the originator of this field of ideas (the recent literature on slow functions, which mostly omits the restriction of monotony, usually fails to mention Pringsheim).

Let $L(x)$ be a function satisfying the above conditions, $L(\theta x) \sim L(x)$ and $L(x) \to \infty$, and let $\sum a(n)/n^s$ be a real Dirichlet series which is convergent for $s > 0$. Then, if $x \to \infty$ and $0 < s \to 0$ respectively, the first of the relations

$$(57_1) \qquad \sum_{n=1}^{\exp x} a(n) \sim L(x); \qquad\qquad (57_2) \qquad \sum_{n=1}^{\infty} a(n)/n^s \sim L(1/s)$$

is known to imply the second as an Abelian consequence. In fact, such an Abelian inference (for power series) was the principal object of Pringsheim's paper. The converse inference, that is, the transition from (57_2) to (57_1), is illegitimate, of course. However, the Tauberian methods of Hardy and Littlewood ([15], footnote on p. 129) prove the truth of the converse inference in case of *non-negative* coefficients $a(n)$. A simple proof of this Tauberian theorem is due to Karamata [22].

Thus (57_1) is equivalent to (57_2) if $a(n) \geqq 0$. But if $a(n) \geqq 0$, then the series (57_2) represents a monotone function, and so the function $L(1/s)$ occurring in (57_2), a function which may be chosen to be identical with the sum of the series, can be assumed to be *monotone*. Since $L(x) \to \infty$, the assumption $L(\theta x) \sim L(x)$ then is equivalent to the estimate $L(x) = O(x^\epsilon)$, where $\theta > 0$ and $\epsilon > 0$ are arbitrary. Accordingly, if $a(n) \geqq 0$, and if $L(x)$ satisfies $L(x) \to \infty$ and $L(x) = O(x^\epsilon)$, then (57_1) is equivalent to (57_2).

In this formulation, no explicit convergence assumption is needed for the series (57_2), if (57_1) is the assumption. In fact, if the sum on the left of (57_1) is called $L(x)$, then (57_1) is true, and so the assumption of the estimate $L(x) = O(x^\epsilon)$ in (57_1) implies that the n-th partial sum of $\sum a(n)$ is $O(n^\epsilon)$. But this necessitates the convergence of $\sum a(n)/n^s$ for every $s > 0$. On the other hand, the divergence of the latter series at $s = 0$ is equivalent to the assumption $L(x) \to \infty$. Consequently, the assumptions under which (57_1) was seen to be equivalent to (57_2) are equivalent to the assumptions italicized before (56). Finally, (56) follows by eliminating L between (57_1) and (57_2).

32. It is clear from Euler's factorization that the two Dirichlet series (45), (41) belonging to the same R have the same abscissa of convergence, i.e., that any prime set, R, has the same convergence exponent as the completely multiplicative set, R^*, generated by R. It is also seen from Euler's factorization that, if λ_R denotes this common exponent of convergence, and if r and r^* run through R

and R^* respectively, then

$$(58) \qquad \sum r^{-\lambda_R} \lesseqgtr \infty \quad \text{according as} \quad \sum (r^*)^{-\lambda_R} \lesseqgtr \infty,$$

provided that R contains an infinity of primes (which certainly is the case if $\lambda_R \neq 0$). If $s > \lambda_R$, it is clear from (53_1), (53_2) that

$$(59_1) \qquad \omega_R(s) = \int_1^{\infty} u^{-s}\, d\pi_R(u); \qquad (59_2) \qquad \zeta_R(s) = \int_1^{\infty} u^{-s}\, d[u]_R,$$

since (45), (41) may be written as

$$(60_1) \qquad \omega_R(s) = \sum R(n)/n^s; \qquad (60_2) \qquad \zeta_R(s) = \sum R^*(n)/n^s.$$

If $\lambda_R \neq 0$, then the series $\omega_R(n\lambda_R)$ converges for every $n > 1$ and tends exponentially to 0 as $n \to \infty$. It follows therefore from (50_1) that there exists a constant c_R satisfying

$$(61) \qquad \log \zeta_R(s) = \omega_R(s) + c_R + o(1) \quad \text{as} \quad s \to \lambda_R, \quad \text{if} \quad \lambda_R > 0,$$

where $s > \lambda_R$. In particular

$$(62) \quad \log \zeta_R(\lambda_R + s) \sim \omega_R(\lambda_R + s) \quad \text{as} \quad s \to 0, \quad \text{if} \quad \zeta_R(\lambda_R) = \infty, \quad \lambda_R > 0.$$

If $\lambda_R = 0$, it is seen from (50_1) that not even the weakened form, (62), of (61) can be asserted. This is the reason that the case $\lambda_R = 0$ must be excluded in the following theorem:

Let λ_R denote the convergence exponent of a prime set R and suppose that λ_R is positive. Then the Dirichlet series $\zeta_R(s)$ has a positive convergence abscissa, λ_R, and the sum of the $-\lambda_R$-th powers of all primes contained in R is convergent or divergent according as $\zeta_R(\lambda_R) < \infty$ or $\zeta_R(\lambda_R) = \infty$. In either case,

$$(63) \qquad \log \int_1^x u^{-\lambda_R}\, d[u]_R \sim \int_1^x -\log(1 - u^{-\lambda_R})\, d\pi_R(u)$$

as $x \to \infty$. In particular

$$(64) \qquad \log \int_1^x u^{-\lambda_R}\, d[u]_R \sim \int_1^x u^{-\lambda_R}\, d\pi_R(u), \quad \text{if} \quad \zeta_R(\lambda_R) = \infty.$$

The asymptotic relation (63) has the same formal structure as Euler's factorization; cf. (53_1), (53_2) and (41). If $\zeta_R(\lambda_R) < \infty$, then (63) cannot be simplified to (64).

Suppose first that $\zeta_R(\lambda_R) = \infty$. Then ($60_1$), ($60_2$) and (58) show that the assumptions of the italicized statement of §31 are satisfied by both Dirichlet series $\sum a(n)/n^s$ defined by $\sum a(n)/n^s = \omega_R(\lambda_R + s)$ and $\sum a(n)/n^s = \zeta_R(\lambda_R + s)$. Thus (56) is applicable to both of these series and supplies, in view of (60_1), (60_2) and (53_1), (53_2), the asymptotic mates

$$(65) \qquad \int_1^{\exp(1/s)} u^{-\lambda_R}\, d\pi_R(u) \sim \omega_R(\lambda_R + s), \qquad \int_1^{\exp(1/s)} u^{-\lambda_R}\, d[u]_R \sim \zeta_R(\lambda_R + s),$$

where $0 < s \to 0$. Clearly, (64) follows from (65) and (62) by placing $x = \exp(1/s)$.

Next, suppose that $\zeta_R(\lambda_R) < \infty$. Then (41) and (45) show that (50_1) is valid for $s = \lambda_R$ and is, as each of the series $\omega_R(\lambda_R)$, $\omega_R(2\lambda_R)$, $\cdots$ which compose $\log \zeta_R(\lambda_R)$, a convergent series with positive terms. But it is clear from (59_1), (59_2) and (53_1), (53_2) that, if $\zeta_R(\lambda_R) < \infty$, then (63) claims just the validity of the expansion (50_1) for $s = \lambda_R$.

Accordingly, all that remains to be shown is that (63) holds in the case $\zeta_R(\lambda_R) = \infty$ also. Hence it is sufficient to verify that (63) is equivalent to (64), if $\zeta_R(\lambda_R) = \infty$. But the truth of this equivalence is clear from (62).

33. For the case $\lambda_R \neq 0$, the asymptotic relation (63) just proved contains about everything that can be said of the asymptotic distribution of the primes in an *arbitrary* completely multiplicative set R^*. Corresponding to the unrestricted nature of an arbitrary prime set R satisfying $\lambda_R \neq 0$, the proof of (63) depended only on the *elementary* Tauberian theorem of §31. If the prime set R is sufficiently regular, non-elementary Tauberian theorems may become applicable, supplying asymptotic relations much sharper than (63).

For instance, if R is the set of all primes, then $\lambda_R = 1$ and $[u]_R = [u]$, and so (64) is reduced to the assertion that the sum of the reciprocal values of all those primes which do not exceed x is asymptotically equal to $\log \log x$. But even Mertens' elementary approximation to the prime number theorem implies that the remainder term of $\log \log x$, instead of being just $o(\log \log x)$, tends to a finite limit as $x \to \infty$. On the other hand, Mertens' approach could not have been applied in case of an arbitrary R (not even if $\lambda_R = 1$ is assumed), since no analogue of Stirling's formula is available for the case in which the sequence of the logarithms of all positive integers is replaced by its subsequence belonging to an unspecified completely multiplicative set, R^*, of positive integers.

An interpretation of (63), where $\lambda_R \neq 0$, is seen from the remarks made in §29 with regard to the analogy of infinitesimal generators.

Another, though related, interpretation results if, corresponding to the Euler factorization (41), the general asymptotic formula (63), where $\lambda_R \neq 0$, is thought of as a manifestation of the statistical independence of the "factors", r, of the "product sequence", R^*. In particular, the actual content of the relation (64) is that the geometrical mean of $x^{-\lambda_R}$ with reference to the Stieltjes weight function (53_2) is asymptotically equal to the arithmetical mean with reference to (53_1).

The statistical independence exhibited by (63) is paralleled by the linear independence of the logarithms of the primes r occurring in R. However, it is hardly possible to establish (63), where $\lambda_R \neq 0$, on the basis of the Kronecker-Weyl theorem alone, since R contains an infinity of primes r.

34. The assumption $\lambda_R > 0$, which, via (62), was essential for both (63) and (64), was not used in the proof of (65). In fact, all that was needed in the proof

of (65) was the divergence of both series $\omega_R(\lambda_R)$, $\zeta_R(\lambda_R)$. But if $\lambda_R = 0$, then both of these series are divergent whenever the set R contains an infinity of primes, since $\omega_R(0)$ is the number of primes contained in R, while $\zeta_R(0) \geqq \omega_R(0)$; cf. (45) and (41). It follows therefore from (65), by placing $\lambda_R = 0$ and $\exp(1/s) = x$, that

$$(66) \qquad \pi_R(x) \sim \omega_R(1/\log x) \quad \text{and} \quad [x]_R \sim \zeta_R(1/\log x) \quad \text{if} \quad \lambda_R = 0,$$

unless the prime set R is a finite set.

In order to illustrate the situation, choose R so that $\lambda_R = 0$ and that there exists a positive index $\alpha = \alpha_R$ satisfying $\omega_R(s) \sim s^{-\alpha}$ as $s \rightarrow 0$. Then $\omega_R(ns) \sim (ns)^{-\alpha}$, if n is fixed, and it is easily justified (cf. Hardy and Ramanujan [18], pp. 251–252) that, if $\omega_R(ns)$ in the infinite sum (50_1) is replaced by $(ns)^{-\alpha}$, the resulting asymptotic relation is correct; so that $\log \zeta_R(s) \sim \zeta(\alpha + 1)s^{-\alpha}$, where $\zeta(\alpha + 1) = \sum n^{-\alpha}/n$. Since this means that $\log \zeta_R(s) \sim \zeta(\alpha + 1)\omega_R(s)$ as $s \rightarrow 0$, it follows from (66), where $x \rightarrow \infty$, that $\log [x]_R \sim \zeta(\alpha + 1)\pi_R(x)$. But (64) would give $\log [x]_R \sim \pi_R(x)$, since $\lambda_R = 0$.

35. If the assumption $\lambda_R = 0$ is retained but the preceding case, where $\log \zeta_R(s) \sim Cs^{-\alpha}$ holds for positive constants C, α, is replaced by $\zeta_R(s) \sim Cs^{-\alpha}$, then α must be an integer and R a finite prime set. In order to see this, it is sufficient to observe that, if R consists of a finite number, j, of primes $r_1, \cdots, r_j$, then obviously

$$(67) \qquad \zeta_R(s) = \prod_{i=1}^{j} (1 - r_i^{-s})^{-1} \sim C_R s^{-j}$$

as $s \rightarrow 0$, where C_R denotes the constant

$$(68) \qquad C_R = \prod_{i=1}^{j} (\log r_i)^{-1}.$$

If the asymptotic relation (67) is compared with (53_2) and (60_2), the general elementary lemma of Hardy and Littlewood, referred to in §31, supplies the Tauberian consequence

$$(69) \qquad [x]_R \sim C_R(\log x)^j/\Gamma(j + 1); \qquad x \rightarrow \infty.$$

Thus (69) is the substitute for (66) in the case of a finite R, a case excluded in (66).

A relation of the form (69) was found in the diary of Gauss [11] (the numerical value of his C_R contains a *lapsus calami*). The first published statement and proof of a relation of the form (69) are given in a paper of Gram [12]; his proof, instead of being Tauberian, depends on a straightforward counting. For more recent references and for finer results and problems, cf. Pillai [32].

The relation (69) may be thought of as a limiting case of (64). Correspondingly, (69) admits of a statistical interpretation similar to, but more explicit than, the one indicated in §33. In order to see this, let R_i denote the prime set

consisting of the single prime r_i; so that R is the logical sum $R_1 + \cdots + R_j$. Since the completely multiplicative set generated by R_i consists of the geometric progression $1, r_i, r_i^2, \cdots$, the number, $[x]_{R_i}$, of its elements not exceeding x is asymptotically proportional to $\log x$, with $1/\log r_i$ as factor of proportionality. Hence, (68) shows that (69) may be written in the form

$$(70) \qquad [x]_R \sim \frac{1}{j!} \prod_{i=1}^{j} [x]_{R_i}, \quad \text{where} \quad R = \sum_{i=1}^{j} R_i .$$

But this is an expression of (asymptotic) statistical independence. In fact, the constant factor, $1/j!$, on the right of the multiplication rule (70) can be thought of as representing the reduction of the $j!$ permutations of the j prime powers (including 0-th powers) the product of which is an arbitrary element of the completely multiplicative set generated by R. The true genesis of this statistical independence can well be observed by proving (69) and (68) in a rather primitive fashion, as follows:

If $j = 1$, then (69) and (68) are obvious. Grant (69) and (68) for a fixed prime set, $R = (r_1, \cdots, r_j)$, and adjoin to this R a $(j + 1)$-th prime, say p. Then it is clear from the definition, (53_2), of the number $[x]_R$, that the number which belongs to the prime set $(r_1, \cdots, r_j, p)$ in the same way as the number $[x]_R$ belongs to $(r_1, \cdots, r_j)$ is identical with

$$\sum_{k=0}^{(\log x)/\log p} [x/p^k]_R, \quad \text{i.e.,} \quad \frac{C_R}{j!} \sum_{k=0}^{(\log x)/\log p} \log^j \frac{x}{p^k} + o(\log^j x) \log (x/p),$$

by (69). Hence, in order to prove (69) and (68) by complete induction, it is sufficient to ascertain that the sum multiplying $C_R/j!$ in the last formula line is asymptotically proportional to $\log^{j+1} x$, with the reciprocal value of $(j + 1) \log p$ as factor of proportionality. But the truth of this asymptotic proportionality is straightforward indeed; it has nothing to do with prime numbers.

It is possible that this approach was the method followed by Gauss [11] (his C_R might have resulted from an erroneous evaluation of what actually is $(j + 1)$ times $\log p$).

It may be mentioned that an obvious adaptation of this complete induction also proves the existence of two positive constants, $a = a_R$ and $b = b_R$, which satisfy the inequalities

$$ax \log^{j-1} x < \int_1^x u \, d[u]_R < bx \log^{j-1} x \quad \text{as} \quad x \to \infty,$$

where, according to (53_2), the integral is identical with the sum of those integers not exceeding x which are contained in the completely multiplicative set generated by a finite prime set $R = (r_1, \cdots, r_j)$.

36. Among Gauss' notes to the *Disquisitiones Arithmeticae*, Schering found the following observation [10]: If Fermat's assertion, according to which all integers exceeding a 2^m-th power of 2 by 1 are primes, were true, the number of

constructible regular polygons having not more than x vertices would be asymptotically proportional to $\log^2 x$, with $\frac{1}{2}/\log^2 2$ as factor of proportionality. This observation, which does not appear to have been checked in the literature, could be treated by the elementary method applied at the end of §35. Another treatment follows by an adaptation of the Tauberian argument applied before (69). This treatment conveniently leads to the asymptotic number of constructible regular polygons without any assumption as to the prime set involved. It depends on an elementary Tauberian theorem which is a straightforward extension (or, rather, corollary) of the one quoted after (57_2) and can, according to Karamata [22], be formulated as follows: If β is a positive index and $L(x)$ denotes a positive function "slow" in the sense that $L(\theta x)/L(x) \to 1$ as $x \to \infty$ holds for every fixed $\theta > 0$, then the two asymptotic relations

$$(71_1) \qquad \sum_{n=1}^{\exp x} a(n) \sim x^{\beta} L(x)/\Gamma(\beta + 1); \qquad (71_2) \qquad \sum_{n=1}^{\infty} a(n)/n^s \sim s^{-\beta} L(1/s),$$

where $x \to \infty$ and $0 < s \to 0$ respectively, are equivalent whenever $a(n) \geqq 0$.

Let $h_m - 1$, where $m = 0, 1, 2, \cdots$, denote the 2^m-th power of 2, and let R denote the set of the primes contained in the sequence $h_0, h_1, \cdots$. By factorizing h_5, Euler disproved Fermat's assertion that every h_m is in R. Since then the character of a few additional integers h_m has been decided. However, it has never been proved that Fermat's assertion is not too false (in the sense that "almost all", or for that matter all but a finite number, of the integers h_m are in R), or that Fermat's assertion is essentially false (in the sense that "almost all," or perhaps all but finite number, of the integers h_m are not in R). In any case, the prime set R determines the set, say G, of those positive integers, say g, for which the regular g-gon is constructible. In fact, the cyclotomic result of the *Disquisitiones Arithmeticae* states that a positive integer is in G if and only if it has either of the forms 2^i, $2^i r_1 r_2 \cdots$, where i is a non-negative integer and $r_1, r_2, \cdots$ are distinct primes contained in R (the 1-gon and the 2-gon are reckoned as constructible). Thus, if $G(n)$ denotes the characteristic function of the set G, then $G(n)$ is the multiplicative function for which $G(p^k)$ is 1 if either the prime p is 2 and $k = 1, 2, \cdots$ or p is in R and $k = 1$, while $G(p^k) = 0$ in the remaining cases. Accordingly, by Euler's factorization,

$$(72) \qquad \sum_{n=1}^{\infty} G(n)/n^s = \sum_{i=0}^{\infty} 2^{-is} \prod_{R} (1 + r^{-s}) \equiv (1 - 2^{-s})^{-1} \zeta_R(s)/\zeta_R(2s),$$

by the product representation (41) of $\zeta_R(s)$. Since every r occurring in the product $\zeta_R(s)$ is an h_m, and since $\log h_m$ is asymptotically proportional to 2^m as $m \to \infty$, it is clear that the Dirichlet series (72) converges for every $s > 0$.

If $0 < s \to 0$, then, since $1 - 2^{-s} \sim s \log 2$, it is seen from (72) that the conditions quoted for the equivalence of (71_1) and (71_2) are satisfied by $a(n) = G(n)$ if there exist an index $\alpha > 0$ and a "slow" function L satisfying

$$(73) \qquad \zeta_R(s)/\zeta_R(2s) \sim s^{-\alpha} L(1/s) \qquad \text{as } s \to 0.$$

In fact, (71_2) then holds for $\beta = \alpha + 1$. But, if $a(n) = G(n)$ and $\beta = \alpha + 1$, elimination of L between (71_1) and (71_2) gives

$$(74) \qquad \Gamma(\alpha + 2) \sum_{n=1}^{x} G(n) \sim \sum_{n=1}^{\infty} G(n)/n^{1/\log x} \qquad \text{as } x \to \infty.$$

Consequently, (74) is true if (73) is true.

In order to discuss the hypothesis (73), consider first the limiting case, represented by Fermat's assertion that every h_m is in R. Then, since $\zeta_R(s)/\zeta_R(2s)$ is $\prod (1 + r^{-s})$ for every R, and since $h_m - 1$ is the 2^m-the power of 2 for $m = 0, 1, \cdots$,

$$\zeta_R(s)/\zeta_R(2s) = \prod_{m=0}^{\infty} \{1 + (2^{2^m} + 1)^{-s}\} \sim \prod_{m=0}^{\infty} \{1 + (2^{2^m})^{-s}\}$$

as $s \to 0$. The last asymptotic relation is readily verified by placing $\theta = 2^{-s}$, taking logarithms and observing that, the number of the integers 2^m which do not exceed x being asymptotically equal to $\log_2 x$, the infinite sum of all the 2^m-th powers of θ, where $0 < \theta < 1$, is asymptotically equal to $\log_2(1 - \theta)^{-1}$ as $\theta \to 1$ (Abel). On the other hand, since every positive integer has exactly one dyadic representation,

$$\prod_{m=0}^{\infty} (1 + \theta^{2^m}) = \sum_{j=0}^{\infty} 1 \cdot \theta^j \equiv (1 - \theta)^{-1}; \qquad 0 < \theta < 1,$$

as easily verified directly (Euler).

It is seen from the last two formula lines, where $\theta = 2^{-s}$, that $\zeta_R(s)/\zeta_R(2s)$ is asymptotically equal to $(1 - 2^{-s})^{-1} \sim (s \log 2)^{-1}$ as $s \to 0$. Hence, (73) is satisfied by $\alpha = 1$ and $L \equiv (\log 2)^{-1}$. Consequently, (74) is applicable. But $\sum G(n)/n^s \sim (s \log 2)^{-1}(s \log 2)^{-1}$, by (72); so that (74), where $\Gamma(\alpha + 2) = \Gamma(3) = 2$, is reduced to

$$\sum_{n=1}^{x} G(n) \sim (2 \log^2 2)^{-1} \log^2 x \quad \text{as} \quad x \to \infty.$$

This asymptotic relation proves the observation of Gauss, quoted above.

If a finite number, say j, of the factors of the above dyadic product are omitted, then, since each of the factors tends to 2 as $\theta \to 1$, the remaining product, instead of being equal to $(1 - \theta)^{-1}$, is asymptotically equal to $2^{-j}(1 - \theta)^{-1}$ as $\theta \to 1$. This means that, if there exists only a finite number, j, of integers $h_0, h_1, \cdots$ which are not primes, then the expression on the right of the last formula line must be multiplied by the constant 2^{-j}. Hence, it is seen by letting $j \to \infty$, that the number of constructible regular polygons having not more than x vertices is $o(\log^2 x)$ or asymptotically proportional to $\log^2 x$ according as the truth of Fermat's assertion is or is not violated by an infinity of the integers h_m.

Consider finally the extreme possibility that Fermat's assertion is violated by all but a finite number, say j, of the integers h_m. Then $\zeta_R(s)$ is of the form (67). Hence, (73) is satisfied by $\alpha = 0$ and $L \equiv 2^j$. Consequently, (74) is applicable and shows (since (72) now gives $\sum G(n)/n^s \sim (s \log 2)^{-1} 2^j$ as $s \to 0$),

that the expression on the right of the last formula line must be replaced by $(\log 2)^{-1}2^j \log x$.

37. The function $\zeta_R(s)/\zeta_R(2s)$ is significant for any prime set R, and not only for the unique R considered in §36. In order to see this, let R^0 denote, for any prime set R, the set of the square-free integers contained in the completely multiplicative set R^*. Then R^0 is a multiplicative set and, if $R^0(n)$ denotes its characteristic function,

$$(75)\qquad \zeta_R(s)/\zeta_R(2s) = \sum R^0(n)/n^s, \quad \text{since} \quad \zeta_R(s)/\zeta_R(2s) = \prod (1 + r^{-s}),$$

by (41), where $\sigma > 1$; cf. (11) and (60_2). Let $\{x\}_R$ denote the number of those integers contained in R^0 which do not exceed x; so that, corresponding to (53_2),

$$(76)\qquad \{x\}_R = \sum_{n=1}^{x} R^0(n).$$

It will now be verified that, if Q denotes the prime set complementary to R, the asymptotic formula (55) can be replaced by the inequality

$$(77)\qquad |\,[x]_R - x/\zeta_Q(1)\,| \leqq \{x\}_Q + x\int_x^\infty u^{-1}\,d[u]_Q\,.$$

Of course, the integral on the right of (77) may be ∞. In fact, (76) shows that this will be the case if and only if $\sum Q^0(n)/n = \infty$. This condition obviously is equivalent to $\sum q^{-1} = \infty$, and therefore to the case (55 bis) of (55). Thus it will be sufficient to verify (77) for the case $1/\zeta_Q(1) \neq 0$, i.e., $\sum Q^0(n)/n < \infty$.

Since $\mu(n)$ is ± 1 or 0 according as n is or is not square-free, (40) implies that $|\,\mu_R(n)| \leqq R^0(n)$ for every n. Hence, if R is replaced by Q both in this inequality and in (43), it follows that $|f'(n)| \leqq Q^0(n)$ when $f(n) = R^*(n)$. On the other hand, from (27) and (11),

$$\left|\sum_{n=1}^{x} f(n) - x\sum f'(n)/n\right| \leqq \sum_{n=1}^{x} |f'(n)| + x\sum_{x+1}^{\infty} |f'(n)|/n,$$

if $\sum |f'(n)|/n < \infty$; in which case (viii bis) and (i) show that $M(f)$ exists and equals $\sum f'(n)/n$. Since $f(n) = R^*(n)$, $|f'(n)| \leqq Q^0(n)$, and $\sum Q^0(n)/n < \infty$ by assumption, it is seen from (53_2), (55) and from the case $R = Q$ of (76), that the proof of (77) is complete.

Another connection between (53_2) and (76) may be formulated as follows:

$$(78)\qquad \log \int_1^x u^{-\lambda_R}\,d[u]_R \sim \log \int_1^x u^{-\lambda_R}\,d\{u\}_R \quad \text{if} \quad \lambda_R \neq 0,$$

where λ_R denotes the convergence exponent of the prime set R or, what is the same thing, the abscissa of convergence of the Dirichlet series (41) or (75). In order to prove (78), it is sufficient to observe that the proof of the italicized result of §32 remains unaltered if (53_2) is replaced by (76). Thus (78) follows from (63) and from the relation which results by replacing $[u]_R$ in (63) by $\{u\}_R$.

The *logarithmic* relation (78) cannot be improved, unless the arbitrary prime set R is subjected to particular assumptions. Such an assumption is the existence of a non-negative index, β, and of a "slow" function, L, satisfying

$$(79) \qquad \zeta_R(\lambda_R + s) \sim s^{-\beta} L(1/s) \qquad \text{as } s \to 0$$

($s > 0$). In fact, (78) can then be improved to

$$(80) \qquad \int_1^x u^{-\lambda_R}\, d[u]_R \sim \zeta_R(2\lambda_R) \int_1^x u^{-\lambda_R}\, d\{u\}_R \quad \text{if} \quad \lambda_R \neq 0.$$

In order to see this, suppose first that $\beta > 0$ in (79). Then the assumptions quoted for the equivalence of (71_1) and (71_2) are satisfied by $\sum a(n)/n^s = \zeta_R(\lambda_R + s)$, where $a(n) = R^*(n)/n^{\lambda_R}$, by (60_2). But elimination of L between (71_1) and (71_2) gives

$$(81) \qquad \Gamma(\beta + 1) \int_1^x u^{-\lambda_R}\, d[u]_R \sim \zeta_R(\lambda_R + 1/\log x),$$

if $R^*(n)/n^{\lambda_R}$ is substituted for $a(n)$ and then the definition (53_2) is applied. If $\beta = 0$ in (79), then (81) follows by using the equivalence of (57_1) and (57_2) instead of the equivalence of (71_1) and (71_2).

Since the Dirichlet series $\zeta_R(s)$ converges for $s > \lambda_R$, it is clear that, if $\lambda_R > 0$, the asymptotic relation (79) remains correct if the expressions on its left and on its right are divided by $\zeta_R(2\lambda_R + 2s)$ and $\zeta_R(2\lambda_R)$ respectively. Hence it is seen from (75) and (76) by a repetition of the proof of (81), that (81) remains correct if $[u]_R$ on the left of (81) is replaced by $\{u\}_R$ and the function on the right of (81) is divided by the constant $\zeta_R(2\lambda_R)$. This variant of (81), when combined with (81) itself, completes the proof of (80).

38. The object of the prime number theorem is the transition from the function (53_2) to the order of the function (53_1) if* $R = P$, where $[x]_R \sim x$ and $\pi_R(x) \sim x/\log x$. The machinery of the prime number theorem will now be used to obtain the converse inference, i.e., the passage from (53_1) to (53_2), if R is random in the sense of §25. It will be shown that, *for a random prime set R, the asymptotic distribution of the integers contained in the completely multiplicative set R^* is given by*

$$(82) \qquad [x]_R \sim \Gamma(\tfrac{1}{2})^{-1} x/\log^{\frac{1}{2}} x; \qquad \text{while } \pi_R(x) \sim \tfrac{1}{2} x/\log x$$

is clear from Borel's form of the law of large numbers, since $\pi_R(x) + \pi_Q(x) = \pi_P(x)$ and $\pi_P(x) \sim x/\log x$.

* According to A. Beurling [2], the function (53_1) remains asymptotically equal to $x/\log x$ if R is such that the function (53_2), instead of being $x + O(1)$, is asymptotically proportional to x with an error term $O(x/\log^c x)$, where $2c > 3$. Incidentally, the considerations of Beurling do not in themselves seem to imply his claim, based on *function-theoretical* reasons, that there exists a *prime set* R which proves that the inequality $2c > 3$ is incapable of improvement.

In this connection, cf. the end of §40 below.

If R is random, the first of the italicized results of §26 implies that $\zeta_R(s)/(s-1)^{\frac{1}{2}}$ is regular on the line $\sigma = 1$; in fact, $(s-1)\zeta(s) \to 1$ as $s \to 1$. But, if the latter relation is combined with (54), it is seen from Borel's form of the law of large numbers that, R being random, $(s-1)^{\frac{1}{2}}\zeta_R(s) \to 1$ as $s \to 1$. Consequently, if R is random, the difference $\zeta_R(s) - (s-1)^{-\frac{1}{2}}$ is regular on the line $\sigma = 1$. It follows therefore from a straightforward extension of Ikehara's theorem (cf. Wintner [44]), that the sum of the first $[x]$ coefficients of the ordinary Dirichlet series $\zeta_R(s)$ is asymptotically proportional to $x/\log^{\frac{1}{2}} x$, with $1/\Gamma(\frac{1}{2}) = \pi^{-\frac{1}{2}}$ as factor of proportionality. This, when compared with (60_2) and (53_2), completes the proof of (82).

39. For any R, let $\zeta_R'/\zeta_R(s)$ denote the logarithmic derivative of $\zeta_R(s)$, and let the function $\Lambda_R(n)$ of the positive integer n be defined by placing

(83) $$-\zeta_R'/\zeta_R(s) = \sum \Lambda_R(n)/n^s.$$

Then it is seen from the product (41) that

(84) $$\Lambda_R(n) \text{ is } 0 \text{ if } n \neq r^k \text{ and } \log r \text{ if } n = r^k,$$

where r and k respectively run through all primes contained in R and through all positive integers. Thus, corresponding to (40),

(85) $$\Lambda_R(n) = \Lambda(n)R^*(n),$$

where $\Lambda(n)$ denotes the function defined by (9_1) or (9_2); that is, the function which results by placing $R = P$, i.e. $r = p$, in (84).

It is clear from (85) that the Dirichlet series (83) and (45) have the same abscissa of convergence, λ_R. Hence, it is easy to see that, *for every prime set* R,

(86) $$\sum_{n=1}^{x} \Lambda_R(n)/n^{\lambda_R} \sim -\zeta_R'/\zeta_R(\lambda_R + 1/\log x) \quad \text{as} \quad x \to \infty$$

(*whether* $\lambda_R > 0$ *or* $\lambda_R = 0$). In fact, (84) and (83) show that the assumptions italicized before (56) are satisfied by $\sum a(n)/n^s = -\zeta_R'/\zeta_R(\lambda_R + s)$, and so (86) follows by placing $\exp(1/s) = x$ in (56).

40. It is clear from (84) that, as $x \to \infty$,

$$\sum_{n=1}^{x} \Lambda_R(n)/n^{\lambda_R} - \sum_{r \leq x} r^{-\lambda_R} \log r \to \sum_{k=2}^{\infty} \sum_{R} r^{-k\lambda_R} \log r < \infty,$$

if $\lambda_R \neq 0$. It follows therefore from (86) and (53_2) that, *if the series* (83), *where* $s > \lambda_R$, *is divergent for* $s = \lambda_R$, *then*

(87) $$\int_1^x u^{-\lambda_R} \log u \, d[u]_R \sim -\zeta_R'/\zeta_R(\lambda_R + 1/\log x), \quad \text{if} \quad \lambda_R \neq 0.$$

This property of an unspecified R is only of the depth of (78) or (63), (64), where again $\lambda_R \neq 0$. Correspondingly, the remarks made at the beginning of §33 apply to (87) also.

For instance, if R is the set, P, of all primes, then $\lambda_R = 1$ and $[u]_R = [u]$; so that, since $-\zeta'/\zeta(s) \sim (s-1)^{-1}$ as $s \to 1$, all that is supplied by the *general* rule (87) is that $\sum (\log p)/p \sim \log x$ if $p \leqq x$. But Mertens' approximation to the prime number theorem improves the remainder term, $o(\log x)$, of this asymptotic relation to $O(1)$, and the prime number theorem even to const. $+o(1)$. However, no corresponding improvements of (87) are possible, unless the choice of the prime set R is subjected to specific conditions of regularity. The necessity of such restrictions is shown by counter-examples which can readily be constructed from a perusal of the proof of (55) and (55 bis), that is, of the sieve of Eratosthenes.

Accordingly, it was essential indeed that in Chap. II the generalization of the prime number theorem to the case of an *arbitrary* prime set R was chosen so as to correspond to the formulation of the prime number theorem in terms of Möbius' function, μ. In fact, the preceding remarks imply that the standard formulation of the prime number theorem, a formulation based on Chebyshev's function, Λ, (or, for that matter, on any function which, as Λ, involves the primes *explicitly*), is incapable of a generalization to the case of an *arbitrary* prime set R.

CHAPTER IV

THE DIVISOR PROBLEMS

41. It is clear from these remarks that what in §18 was fundamental for the extension of the prime number theorem to the case of an arbitrary prime set R was the *multiplicative* character of the function $\mu_R(n)$ of n; a function which can attain only the values ± 1 and 0. This observation naturally suggests a question representing a generalization of the problem answered by the italicized result of §18.

In fact, this result implies, by the case $a(m) = \mu_R(m)/m$ of (24), that (12) remains valid if $\mu = \mu_P$ is replaced by any μ_R, i.e., that $M(\mu_R)$ exists and is 0 for every prime set R. Conversely, this implies the italicized result of §18, if use is made of the case $f'(n) = \mu_R(n)$ of the elementary fact (v bis) and of (55), (55 bis). Every $\mu_R(n)$ is a multiplicative function attaining only the values ± 1 and 0. Those values 0 which are attained for not square-free values of n are harmless from the analytical point of view, since the sum of the reciprocal values of all but the first powers of all the primes is convergent. However, $\mu_R(n)$ will in general attain the value 0 on a set of primes for which the sum of the reciprocal values is divergent.

In addition, no $\mu_R(n)$ can become $+1$ when n is a prime. It is precisely this fact that makes possible the convergence of every series $\sum \mu_R(n)/n$, the series $\sum +1/n$ being divergent. Thus there arises the question as to whether or not $M(f)$ must exist for *every* real-valued multiplicative function of constant absolute value.

It will turn out (in §44) that the answer to this question is affirmative. The proof will involve, in a curious fashion, that generalization of Dirichlet's divisor problem which results if the set of all positive integers is replaced by an arbitrary completely multiplicative set.

It is instructive to consider the announced generalization of the italicized result of §18 from the following point of view: If $f(n)$ is an *arbitrary* function attaining only the values ± 1, then, according to Borel's form of the law of the large numbers, the mean $M(f)$ will exist (and be 0) for *almost all* of the functions $f(n)$, where the excluded set of measure 0 refers to the $(\pm, \pm, \cdots)$-space of §24. But the exceptional set of those selections for which $M(f)$ fails to exist is not vacuous; in fact, it is a set of the second category on the interval $0 \leqq \xi \leqq 1$ on which the whole of the $(\pm, \pm, \cdots)$-space is mapped by the dyadic expansion of ξ. However, the result to be proved states that no point of this set of second category will represent a *multiplicative* function $f(n)$. This is the more interesting as the set of those points of the interval $0 \leqq \xi \leqq 1$ which represent *multiplicative* functions $f(n)$ is readily seen to be of the second category.

42. For any prime set R, define the function $\tau_R(n)$ by

$$\sum \tau_R(n)/n^s = \prod (1 - r^{-s})^{-2} \equiv \zeta_R(s)^2; \tag{88}$$

cf. (11), (41). Clearly, $\tau_R(n)$ is that multiplicative function for which $\tau_R(p^k)$ is $k + 1$ or 0 according as the prime, p, is or is not a prime, r, contained in R. In particular, if $R = P$, so that $\zeta_R(s)^2 = \zeta(s)^2$, then $\tau_R(n) = \tau(n)$, if $\tau(n)$ denotes, as in (5), the number of all divisors of n. It follows that, corresponding to (40) and (85),

$$\tau_R(n) = \tau(n)R^*(n). \tag{89}$$

In fact, (89) is true if $n = p^k$, since $R^*(p^k)$ is 1 or 0 according as p is or is not an r. But this proves (89) for every n, since all three functions τ_R, τ, R^* are multiplicative.

Clearly, (18) means that the Dirichlet series $\sum c(n)/n^s$ is the product of $\sum a(n)/n^s$ and $\sum b(n)/n^s$. Hence, if $a(n) = R^*(n)$ and $b(n) = R^*(n)$, then $c(n) = \tau_R(n)$, by (60_2) and (88). Consequently, if

$$T_R(x) = \sum_{n=1}^{x} \tau_R(n), \tag{90}$$

then $C(x) = T_R(x)$ in (19), while $A(x) = [x]_R$ and $B(x) = [x]_R$, by (53_2). Accordingly, if t in (20) is chosen to be $x^{\frac{1}{2}}$, then

$$T_R(x) = 2 \sum_{n=1}^{x^{\frac{1}{2}}} R^*(n)[x/n]_R - [x^{\frac{1}{2}}]_R^2 . \tag{91}$$

Incidentally, if t in (20) is chosen to be x, it follows that

$$T_R(x) = \sum_{n=1}^{x} R^*(n)[x/n]_R \equiv \int_1^x [x/u]_R \, d[u]_R . \tag{91 bis}$$

For every R, let $\rho_R(x)$ be defined by

$$[x]_R = \zeta_Q(1)^{-1}x + \rho_R(x); \qquad \text{so that } \rho_R(x) = o(x), \tag{92}$$

by (55), where, according to (55 bis), it is allowed that $\zeta_Q(1)^{-1} = 0$, i.e., that $\sum q^{-1} = \infty$. It is understood that q runs through all primes contained in the complement, $Q = P - R$, of R.

If (92) is substituted into (91), it is seen that $T_R(x)$ is identical with the sum of

$$2\zeta_Q(1)^{-1}x \sum_{n=1}^{x^{\frac{1}{2}}} R^*(n)/n + 2 \sum_{n=1}^{x^{\frac{1}{2}}} R^*(n)\rho_R(x/n) - \zeta_Q(1)^{-2}x$$

and of the two additional terms $-2\zeta_Q(1)^{-1}x^{\frac{1}{2}}\rho_R(x^{\frac{1}{2}})$, $-\rho_R(x^{\frac{1}{2}})^2$. The latter are $o(x)$, since $\rho_R(x) = o(x)$, by (92). In addition, the sum preceding the term $-\zeta_Q(1)^{-2}x$ in the last formula line is $o(x)$, since the characteristic function

$R^*(n)$ is either 1 or 0 and

$$\sum_{n=1}^{x^{\frac{1}{2}}} | \rho_R(x/n) | = \sum_{n=1}^{x^{\frac{1}{2}}} o(x/n) = \sum_{n=1}^{x^{\frac{1}{2}}} o(x/x^{\frac{1}{2}}) = O(x^{\frac{1}{2}})o(x^{\frac{1}{2}}).$$

Finally, $\sum_{n=1}^{x^{\frac{1}{2}}} R^*(n)/n = \int_1^{x^{\frac{1}{2}}} u^{-1}\, d[u]_R$, by ($53_2$). Consequently,

$$T_R(x) = 2\zeta_Q(1)^{-1}x \int_1^{x^{\frac{1}{2}}} u^{-1}\, d[u]_R - \zeta_Q(1)^{-2}x + o(x). \tag{93}$$

43. It will now be easy to prove the following theorem:

(I) *The function $T_R(x)$ defined by (90) is of the form $o(x)$ or is asymptotically proportional to $x \log x$, with the positive number $M(R^*)^2$ as factor of proportionality, according as the measure, $M(R^*) = \zeta_Q(1)^{-1}$, of the completely multiplicative set, R^*, is or is not 0, i.e., according as the sum, $\sum q^{-1}$, of the reciprocal values of the primes, q, contained in the complement, $Q = P - R$, of the prime set, R, is divergent or convergent.*

The sharpness of the alternative expressed by (I) is unexpected, since, if its first case, just as in the proof of (55 bis), is thought of as being the limiting case of the second, then all that follows for the former is $T_R(x) = o(x \log x)$. In other words, (I) implies that $T_R(x) = o(x)$ *whenever* $T_R(x) = o(x \log x)$. Correspondingly, (I) can be interpreted as an arithmetical counterpart of the celebrated alternatives of Borel [3].

Since (93) is identical with $T_R(x) = o(x)$ when $\zeta_Q(1)^{-1} = 0$, it is sufficient to consider the case $\zeta_Q(1)^{-1} > 0$. But a partial integration shows that, since $[u]_R = \zeta_Q(1)^{-1}u + o(u)$ in view of (92), the integral occurring in (93) can be written in the form

$$O(1) + \int_1^{x^{\frac{1}{2}}} u^{-2}[u]_R\, du = O(1) + \zeta_Q(1)^{-1} \int_1^{x^{\frac{1}{2}}} u^{-1}\, du + \int_1^{x^{\frac{1}{2}}} o(u^{-1})\, du,$$

which is $\zeta_Q(1)^{-1} \log (x^{\frac{1}{2}}) + o(\log x)$. Since this implies, by (93), that $T_R(x) \sim \zeta_Q(1)^{-2}x \log x$, the proof of the italicized statement, (I), is complete.

Clearly, the truth of the sharp alternative of (I) is just a manifestation of the sieve of Eratosthenes. On the other hand, the following theorem is substantially analytical.

(II) *The series $\sum \tau'_R(n)/n$ converges (to 0) whenever the measure, $M(R^*)$, of the completely multiplicative set, R^*, is 0, i.e., whenever the sum of the reciprocal values of the primes not contained in the prime set R is divergent.*

First, from (88) and (10),

$$\sum \tau'_R(n)/n^s = \zeta_R(s)^2/\zeta(s) \equiv \prod (1 - r^{-s})^{-1} \prod (1 - q^{-s}), \tag{94}$$

by (54), where $\sigma > 1$. Since $(1 - r^{-s})^{-1} = 1 + r^{-s} + r^{-2s} + \cdots$, it is clear from (94) that $\tau'_R(n)$ is a multiplicative function which can attain only the values ± 1 and 0. In particular, $\tau'_R(n) = O(1)$. It follows therefore from (vi)

that the series $\sum \tau_R'(n)/n$ converges if and only if $M(\tau_R)$ exists; in which case the sum of the series must be $M(\tau_R)$, by (i). Since (90) and (14) show that the first case of the alternative (I) is identical with the existence and the vanishing of $M(\tau_R)$, the proof of (II) is complete. (Incidentally, (I) implies that $M(\tau_R)$ must be 0, if it exists at all).

The content of (II) is substantially equivalent to the prime number theorem. For, on the one hand, the difficult part of (vi), used in the proof of (II), is not deeper than the prime number theorem (cf. the beginning of §6). And, on the other hand, (II) will now be shown to imply the result announced in §41; a result which, as explained in §41, contains the generalized prime number theorem of §18.

44. To this end, the following elementary fact will be needed:

LEMMA. *If $a(n)$ and $b(n)$ are two functions for which $\sum a(n)/n$ converges absolutely and $M(b)$ exists, and if $c(n)$ denotes the function defined by* (18), *then $M(c)$ exists and equals $M(b) \sum a(n)/n$.*

This is a $(C, 1)$-variant of the Mertens-Stieltjes theorem on the multiplication of series.

In order to verify the Lemma for the sake of completeness, choose $t = x$ in (20). The resulting identity can be written in the form

$$C(x) = \sum_{n=1}^{x} a(n)M(b)x/n - \sum_{n=1}^{x} a(n)\{M(b)x/n - B(x/n)\},$$

since the terms inserted, that is, those multiplied by the constant $M(b)$, cancel each other. According to (19) and (14), the assertion of the Lemma is that $C(x)/x \to M(b) \sum a(n)/n$ as $x \to \infty$. But the first sum on the right of the preceding representation of $C(x)$ is asymptotically proportional to x, with $M(b) \sum a(n)/n$ as factor of proportionality. Hence, it is sufficient to show that the second sum is $o(x)$. But the existence of $M(b)$ means, again by (19) and (14), that $M(b)x - B(x) = o(x)$. Since $n \leqq x^{\frac{1}{2}}$ implies that $x/n \geqq x^{\frac{1}{2}}$, it follows that the second sum is majorized by

$$\sum_{n=1}^{x^{\frac{1}{2}}} | a(n) | o(x^{\frac{1}{2}}) + \sum_{x^{\frac{1}{2}}}^{x} | a(n) | o(x/n) \equiv o(x^{\frac{1}{2}}) \sum_{n=1}^{x^{\frac{1}{2}}} | a(n) | + O(x) \sum_{x^{\frac{1}{2}}}^{\infty} | a(n) | /n.$$

And this is $o(x)$ in virtue of the absolute convergence of $\sum a(n)/n$. In fact, the sum multiplying $o(x^{\frac{1}{2}})$ is $o(x^{\frac{1}{2}})$, as seen by writing $x^{\frac{1}{2}}$ for n, and $| a(n) | /n$ for $a(n)$, in (24).

It will now be easy to prove the following theorem:

(II*) *If $f(n)$ is any multiplicative function attaining only the values ± 1, then $M(f)$ exists.*

With reference to a given $f(n)$, let the set, P, of all primes, p, be disjoined into two complementary subsets, R and $Q = P - R$, by placing a p into R or into Q according as $f(p) = 1$ or $f(p) = -1$. Then, by Euler's factorization,

$$(95) \qquad \sum f(n)/n^s = \prod (1 + r^{-s} \pm r^{-2s} \pm \cdots) \prod (1 - q^{-s} \pm q^{-2s} \pm \cdots),$$

($s > 1$), where r and q run through R and Q respectively and the sign of every $\pm r^{-ks}$ and of every $\pm q^{-ks}$ depends on the choice of the function $f(n)$ if $k = 2, 3, \cdots$. However, since the sum of the reciprocal values of all but the first powers of all primes (that is, the double series $\sum\sum p^{-k}$, where $k = 2, 3, \cdots$) is convergent, it is clear from the Lemma that, in order to prove (II*), it is sufficient to establish the existence of $M(f)$ for those functions $f(n)$ for which every $\pm r^{-ks}$ is $+r^{-ks}$ and every $\pm q^{-ks}$ is $(-1)^k q^{-ks}$, for instance. This reduces (95) to

$$\text{(95 bis)} \qquad \sum g(n)/n^s = \prod (1 - r^{-s})^{-1} \prod (1 + q^{-s})^{-1},$$

($s > 1$), if $f(n)$ is called $g(n)$ in this particular case. But if all terms $(-1)^k q^{-ks}$ belonging to $k = 2, 3, \cdots$ are omitted in $(1 + q^{-s})^{-1} = 1 - q^{-s} + q^{-2s} - \cdots$, then (95 bis) becomes identical with (94), i.e., $g(n)$ becomes $\tau_R'(n)$. On the other hand, if the proof of the transition from (95) to (95 bis) is repeated, it follows again from the Lemma that, instead of proving the existence of $M(g)$ for every $g(n)$, it is sufficient to prove the existence of $M(\tau_R')$ for every R.

The proof for the existence of $M(\tau_R')$ must naturally distinguish the two cases represented by the alternative of (I). In the first case, where $\sum q^{-1} = \infty$, it follows from (II), by an application of (24) to $a(n) = \tau_R'(n)/n$, that the n-th partial sum of $\sum \tau_R'(n)$ is $o(n)$. This means, by (14), that $M(\tau_R')$ exists (and is 0). In the second case, where $\sum q^{-1} < \infty$, it is seen, by applying the Lemma in exactly the same way as above, that it is sufficient to prove the existence of $M(\tau_R')$ in the particular case in which the product $\prod(1 - q^{-s})$ occurring in (94), instead of being convergent at $s = 1$, is vacuous. But then R becomes the set, P, of all primes, i.e., $\tau_R'(n)$ becomes the function $\tau_P'(n) = \tau'(n)$; a function for which $M(\tau')$ exists and equals 1, by (5).

This completes the proof of (II*).

45. In view of (v), (vi), (vii), the following disjunction of the content of (II*) is not without interest.

(II′) *In* (II*), *all the following possibilities are equivalent*: $M(f)$ *is not* 0; *the series* $\sum f(n)/n$ *diverges*; *the sum of the reciprocal values of those primes for which* f *becomes* -1 *is convergent.*

In fact, (II′) may be obtained from the first part of (III) below in exactly the same manner in which (II*) was deduced from (II).

Remark. *If* $M(f) \neq 0$, *then* $M(f) > 0$ *in* (II′).

In fact, if the Dirichlet series on the left of (95) is multiplied by $(s - 1)$, where $s > 1$, then, according to Dirichlet's Abelian theorem, the resulting product must tend to $M(f)$ as $s \to 1$. However, it is clear that every factor on the right of (95), where $s > 1$, is positive.

The above results admit of a statistical interpretation in terms of an heuristic consideration of Sylvester ([35]; cf. Wintner [45], pp. 14–15). In fact, the content of (II*) and (II′) together may be summarized by saying that, for multiplicative functions attaining only the values ± 1, the result supplied by Sylvester's considerations happens to be correct (such is not the case for arbitrary multiplicative functions; cf. Wintner [45], Theorem IV).

In order to see this, let $f_p(n)$ denote, for every prime p, the multiplicative function defined by

$$(95^*)\qquad \sum_{n=1}^{\infty} f_p(n)/n^s = 1 + \sum_{n=1}^{\infty} f(p^n)/p^{ns}; \qquad \text{so that} \quad f(n) = \prod_p f_p(n)$$

in virtue of (95), the infinite product $f(n) = \prod f_p(n)$ having only a finite number of factors distinct from 1 for every fixed n. In other words, f_p is that multiplicative function of n (attaining only the values ± 1) for which the value $f_p(n)$ attained when n is a prime power is $f(n)$ or $+1$ according as the prime power is or is not a power of the fixed prime p. Hence, it is clear from (4_1)–(4_2), (37) and (viii bis), (i), that $M(f_p)$ exists and is represented by $1 + \sum \{f(p^n) - f(p^{n-1})\}/p^n$ for every fixed p, where $f(p^n) = \pm 1$.

Sylvester's heuristic principle is that $M(f)$ exists and, corresponding to $f(n) = \prod f_p(n)$, is represented by $\prod M(f_p)$, whenever $M(f_p)$ exists for each of the multiplicative functions $f_p(n)$. In the present case, this gives

$$M(f) = \prod_p M(f_p) = \prod_p \left(1 + \sum_{n=1}^{\infty} \{f(p^n) - f(p^{n-1})\}/p^n\right),$$

where $f(p^0) = f(1) = 1$. But $f(p^k) = \pm 1$, and so the last product converges to a positive number or diverges to 0 according as the sum of the reciprocal values of those primes p for which $f(p)$ is -1 is convergent or divergent. Accordingly, what is assured by (II*) and (II′) together is precisely the truth of the last formula.*

It will now be shown that, for straightforward Diophantine reasons, the restriction of the assertion of (II*) to *real-valued* functions is essential indeed.

(II* bis) *If $f(n)$ is a multiplicative function satisfying $|f(n)| = 1$ for every n, then $M(f)$ need not exist.*

In fact, since $\sum 1/p$, where p runs through all primes, is divergent, it is easy to construct a sequence of real numbers λ_p such that $\sum (e^{i\lambda_p} - 1)/p^s$, where $s > 1$, fails to tend to a limit as $s \to 1$. Now define a multiplicative function, $f(n)$, of constant absolute value 1, by assigning $f(p^k) = e^{ik\lambda_p}$ for $k = 1, 2, \cdots$. Then, if $s > 1$, Euler's factorization gives

$$\zeta(s)^{-1} \sum f(n)/n^s \equiv \prod (1 - p^{-s}) \prod (1 - e^{i\lambda_p} p^{-s})^{-1}$$
$$\sim \exp \{\sum (e^{i\lambda_p} - 1)/p^s + \text{const.}\}$$

as $s \to 1$, since $\sum p^{-2} < \infty$. It follows therefore from $(s-1)^{-1} \sim \zeta(s)$ and from the choice of the constants λ_p, that $(s-1) \sum f(n)/n^s$, where $s > 1$, cannot tend to a limit as $s \to 1$. Hence, the existence of $M(f)$ is precluded by Dirichlet's Abelian theorem.

* In view of this situation, it seems to be a reasonable guess that, while $M(f)$ exists, by (II*), for an arbitrary multiplicative function $f(n) = \pm 1$, such a function is almost-periodic (B) *only* in the case $M(f) > 0$; a case which, according to (II′), is of a trivial nature. It is instructive to compare this with the "harmonic law of large numbers" (Wiener and Wintner [41]).

It is seen from the comments made in §41, that the truth of (II*), where $\sum f(n)/n$ *can* be divergent by (II′), is largely due to the sharpness of the alternative pointed out after (I).

46. It was shown at the end of the proof of (II*) that $M(\tau_R')$ exists for every R. This may be amplified as follows:

(III) *If the prime set* R *is arbitrary,* $M(\tau_R')$ *exists, and is or is not* 0 *according as one or none of the following conditions, which are all equivalent, is satisfied:* $\sum \tau_R'(n)/n$ *converges;* $M(\tau_R)$ *exists;* R^* *is of measure* 0, *i.e.,* $\sum q^{-1} = \infty$. *In either case,*

$$T_R(x) = x\sum_{n=1}^{x} \tau_R'(n)/n + (C - 1)M(\tau_R')x + o(x), \tag{96}$$

where C *denotes Euler's constant.*

It is sufficient to prove (96), since the three criteria of (III) for the value of $M(\tau_R')$ then follow from (I), (II) and (90). But (90) also shows that (96) can be written in the form (17), where $f(n) = \tau_R(n)$. Then $f'(n) = \tau_R'(n)$ is either ± 1 or 0, and so the assumptions under which (v bis) assures (17) are satisfied. This proves (III).

In connection with this application of (v bis), it must be emphasized that $\tau_R(n)$ is *not* the function, $\int R^*(n)$, which results if the function occurring in the Remark at the end of §15, where the measurable set, S, is arbitrary, is considered in the particular case of a completely multiplicative set, $S = R^*$. In fact, it is easily verified from (10) that $\zeta_R(s)^2$ on the right of (88) must be replaced by $\zeta(s)\zeta_R(s)$, if the function $\tau_R(s)$ on the left of (88) is replaced $\int R^*(n)$. Correspondingly, the relation (*) at the end of §15 cannot be combined with (96).

47. On the other hand, (93) supplies, precisely in the elementary case excluded in (II), the following criterion:

(IV) *If the measure* $M(R^*)$ *is not* 0, *i.e., if* $\sum q^{-1} < \infty$, *then the general assertion,* $T_R(x) \sim M(R^*)^2\, x \log x$, *of* (I) *can be refined to*

$$T_R(x) = M(R^*)^2\, x \log x + \text{const.}\, x + o(x) \tag{97}$$

if and only if the improper integral

$$C_R = \int_1^{\infty} u^{-2} \rho_R(u)\, du = \lim_{x\to\infty} \int_1^{x} u^{-2} \rho_R(u)\, du \tag{98}$$

exists, where $\rho_R(x)$ *denotes, as in* (92), *the error term of* (55).

In fact, if the integral on the right of (93) is integrated partially, and then (92) is inserted into (93), the assumption that $\zeta_Q(1)^{-1} = M(R^*)$ is not 0 is seen to imply that (93) can be written in the form (97) if and only if the limit (98) exists.

Incidentally, it is clear from (I) and from the proof of (55 bis), that (97) does

not imply the existence of the limit (98), if the assumption, $M(R^*) \neq 0$, of (IV) is omitted.

If (59_2), where $\sigma > 1$, is integrated partially, and then (92), where $\zeta_Q(1)^{-1} = M(R^*)$ may or may not be 0, is substituted, it is seen that

$$\zeta_R(s)/s = M(R^*)/(s-1) + \int_1^\infty u^{-s-1}\rho_R(u)\,du \tag{99}$$

holds for every prime set R. But if R is such as to render the integral (98) convergent, then, by the integral form of Abel's continuity theorem, the integral on the right of (99) must tend to the limit (98) as $s \to 1$, where $s > 1$. Hence, (IV) entails the following corollary:

(IV bis) *If the measure $M(R^*)$ is not 0, i.e., if $\sum q^{-1} < \infty$, then the assumption (97) implies the existence of a constant, C_R, satisfying*

$$\zeta_R(s) - M(R^*)/(s-1) \to C_R \quad \text{as} \quad s \to 1, \text{ if } s > 1. \tag{100}$$

Clearly, (IV bis) remains valid if the restriction of (100) to the half-line $s > 1$ is replaced by the assumption of a Stolz wedge, $|\arg(s-1)| < \frac{1}{2}\theta\pi$, where $0 < \theta < 1$. In view of §22, it must be expected that $\theta = 1$ cannot in general be allowed.

48. The above results on $\tau_R(n)$ do or do not involve the prime number theorem according as R^* is or is not of measure 0. On the other hand, the situation is uniformly elementary if the *multiplicative* function $\tau_R(n)$ is replaced by the *additive* function, say $\nu_R(n)$, which is related to $\tau_R(n)$ in the same way as the two divisor functions, $\tau(n)$ and $\nu(n)$, of §14 correspond to each other in the case $R = P$.

In order to see this, let $\nu_R(n)$ be defined by the assignment $\nu'_R(n) = R(n)$, where $R(n)$ denotes the characteristic function of an arbitrary prime set, R. Then (1) shows that $\nu_R(n)$ is the number of those distinct prime divisors of n which are contained in R. Since there are in R at most $O(x/\log x) = o(x)$ primes not exceeding x, it is clear from $R(n) = O(1)$ that $M(\nu'_R)$ exists and is zero. Hence, the assertion, (17), of (v bis) is applicable to $f(n) = \nu_R(n)$ and states that, as $x \to \infty$, the difference between the ratio $N_R(x)/x$ and the $[x]$-th partial sum of the series $\sum R(n)/n$ tends to 0, where, corresponding to (90),

$$N_R(x) = \sum_{n=1}^{x} \nu_R(n). \tag{90$'$}$$

According to (53_1), this limit relation can be written as

$$N_R(x) = x\int_1^x u^{-1}\,d\pi_R(u) + o(x). \tag{96$'$}$$

It corresponds to (96) and holds, as (96), for any R. In particular, it implies that $M(\nu_R)$ exists if and only if the integral

$$\int_1^\infty u^{-1}\,d\pi_R(u) \tag{98$'$}$$

is convergent. This corresponds to the criterion (98), since, whether $M(\nu_R)$ does or does not exist, a partial integration supplies the identity

$$\int_1^x u^{-1}\, d\pi_R(u) = o(1) + \int_1^x u^{-2}\, \pi_R(u)\, du,$$

$x^{-1}\pi_R(x)$ being at most $O(1/\log x) = o(1)$. This identity also shows that the above integral representation of $N_R(x)$ in the general case can be written in the form

$$N_R(x) = x\int_1^x u^{-2}\, \pi_R(u)\, du + o(x). \tag{93'}$$

In view of (53_1), the former representation of $N_R(x)$ is identical with

$$N_R(x) = x \sum_{r \leqq x} r^{-1} + o(x). \tag{96' bis}$$

In the limiting case of the set of all primes, p, this becomes

$$\sum_{n=1}^{x} \nu(n) = x \log\log x + \text{const.}\, x + o(x),$$

since even Mertens' elementary relation implies that $\sum_{p \leqq x} p^{-1} - \log\log x$ tends to a limit as $x \to \infty$ (cf. Hardy and Ramanujan [18], p. 263).

CONTENT AND MEASURE

49. The existence of the mean $M(f)$ of a function $f(n)$ may be described as follows: There exists a constant, $M(f)$, by means of which the $[x]$-th partial sum of $\sum f(n)$ is representable in the form $M(f)x + o(x)$. Correspondingly, the existence of the logarithmic mean, say $L(f)$, may be defined as follows: There exists a constant, $L(f)$, by means of which the $[x]$-th partial sum of $\sum f(n)/n$ is representable in the form $L(f) \log x + o(\log x)$.

A partial summation shows that, if $M(f)$ exists, then $L(f)$ exists and equals $M(f)$. The converse is not true. In this direction, the following fact is of interest in view of (ii) and (i):

(ii bis) *If $\sum f'(n)/n$ is convergent, then $L(f)$ exists and $\sum f'(n)/n = L(f)$.*

In this Abelian lemma, convergence of the series $\sum f'(n)/n$ cannot be weakened to its (C, ϵ)-summability. In fact, $f(n)/n = o(\log n)$ obviously is a necessary condition for the existence of $L(f)$. Hence, $L(f)$ cannot exist if $f'(n) = (-1)^n n \log n$, since then $| f(p) | = p \log p$ for every prime p, by (1). However, $\sum f'(n)/n = \sum (-1)^n \log n$ is a (C, ϵ)-summable series for every $\epsilon > 0$.

In order to prove (ii bis), let (1) be written in the form

$$f(m)/m = \sum_{d \mid m} f'(d)/m; \quad \text{so that} \quad F(n) = \sum_{m=1}^{n} F^0([n/m])/m,$$

if $F(n)$ and $F^0(n)$ denote the n-th partial sums of $\sum f(n)/n$ and $\sum f'(n)/n$ respectively. Thus

$$F(n)/\log n = \sum_{k=1}^{n} c_{nk} F^0(k), \quad \text{where} \quad c_{nk} \log n = \sum_{n/(k+1) < m \leq n/k} 1/m$$

$(n > 1)$. Clearly, $(c_{n1} + \cdots + c_{nn}) \log n$ is the n-th partial sum of the harmonic series; so that $c_{n1} + \cdots + c_{nn} \to 1$ as $n \to \infty$. It is also clear that $c_{nk} \to 0$, if k is fixed. Since $c_{nk} \geqq 0$, it follows that the matrix transforming $F^0(n)$ into $F(n)/\log n$ satisfies the regularity conditions of Toeplitz [36], i.e., that $F(n)/\log n \to F^0(\infty)$ holds whenever the limit $F^0(\infty)$ exists. But this precisely is the assertion of (ii bis).

50. In §15, a set S was called measurable if its characteristic function, $S(n)$, had a mean, $M(S)$, which then was called the measure of S. In the sequel, the logarithmic mean, $L(S)$, of S will also be considered. Since the existence of $M(S)$ requires of S more than the existence of $L(S)$, let $L(S)$ now be called "measure", and $M(S)$ "content". Measurability will from now on be meant

only in the sense of the logarithmic mean. Correspondingly, the Theorem of §15 must now be restated as follows:

(α) *A set S has a content, $M(S)$, if and only if the Dirichlet series $\sum S'(n)/n^s$ is convergent at the point $s = 1$; in which case $\sum S'(n)/n = M(S)$.*

While (α) depends on more than the prime number theorem, the following criterion is elementary:

(α^*) *A set S has a content, $M(S)$, if and only if $(1 - r) \sum S(n)r^n$, where $r < 1$, tends to a limit as $r \to 1$; in which case the limit is $M(S)$.*

In fact, since $S(n) = O(1)$, the assertion of (α^*) is contained in the Tauberian theorem of Hardy and Littlewood, according to which (A)-summability and (C,1)-summability are equivalent for those series in which the n-th partial sum is $O_L(1)$.

The criteria (α) and (α^*) have analogues for the case in which the measure replaces the content. However, *both* of these analogues are elementary. In fact, what corresponds to (α) is the following criterion:

(β) *A set S has a measure, $L(S)$, if and only if the Dirichlet series $\sum S'(n)/n^s$, where $s > 1$, tends to a limit as $s \to 1$; in which case the limit is $L(S)$.*

Needless to say, $\sum S'(n)/n^s$ is absolutely convergent for $s > 1$ whether the set be measurable or not. In fact, since $S(n) = O(1)$, and since the number of all divisors of n is $O(n^\epsilon)$ for every $\epsilon > 0$, it is clear from (1) that $S'(n) = O(n^\epsilon)$ for every $\epsilon > 0$. Similarly, $\sum S(n)/n^s$ is always absolutely convergent for $s > 1$. Since $\zeta(s) \sim (s - 1)^{-1}$ as $s \to 1$, it is seen from (10) that (β) is equivalent to the following analogue of (α^*):

(β^*) *A set S has a measure, $L(S)$, if and only if $(s - 1) \sum S(n)/n^s$, where $s > 1$, tends to a limit as $s \to 1$; in which case the limit is $L(S)$.*

If $a(n)$ denotes $S(n)/n$ for an arbitrary set S, the italicized assumptions of (56) are satisfied. Hence, if $\exp(1/s) = x$ in (56), it follows that

$$\sum_{n=1}^{x} S(n)/n \sim \sum_{n=1}^{\infty} S(n)/n^{1+1/\log x} \qquad \text{as } x \to \infty$$

holds for *any* set S. In particular, S has a measure if and only if the product of $1/\log x$ and of the infinite series on the right of the last formula tends to a limit, $L(S)$, as $x \to \infty$. Hence, (β^*) follows by placing $1/\log x = s - 1$.

A sufficient condition for the measurability of a set S is that the logarithmic derivative of $\sum S(n)/n^s$ should not exceed that of $\sum 1/n^s = \zeta(s)$, as $s \to 1 + 0$. In fact, (10) then shows that, $\sum S(n)/n^s$ and $\zeta(s)$ being always non-increasing and positive for $s > 1$, the derivative of $\sum S'(n)/n^s$ cannot be positive as $s \to 1$, and so the existence of $L(S)$ is assured by (β).

A comparison of (α) and (β) explains the nature of the difference between content and measure. It also follows that *a set S has a content if and only if it has a measure and satisfies the estimate*

$$\sum_{n=1}^{x} \frac{S'(n)}{n} \log n = o(\log x); \qquad \text{cf. (34).}$$

This is clear from (α) and (β) in view of Schnee's extension (to Dirichlet series) of Tauber's theorem (on power series); cf., e.g., Hardy and Riesz [19], p. 46.

If the pair (α), (β) is replaced by the elementary pair (α^*), (β^*), there *cannot* result an "explicit" criterion corresponding to the last italicized statement. In fact, while the existence of Dirichlet's limit, $\lim\,(s - 1)\sum S(n)/n^s$, where $s > 1$, is implied by that of Frobenius' limit, $\lim\,(1 - r)\sum S(n)r^n$, the truth or falsehood of the converse inference for a given set S involves the behavior of $\sum S(n)/n^s$ for *complex* values of s near the *whole* of the line $\sigma = 1$, as exemplified by Ikehara's theorem. On the other hand, it is seen from §22–§23 that, even if the structure of S is so simple as that of a completely multiplicative set, Ikehara's theorem supplies only a sufficient condition.

An example, proving that there are characteristic functions $S(n)$ for which Dirichlet's limit exists but Frobenius' limit fails to exist, may be obtained by a slight modification of a Dirichlet series considered by Dedekind [5]. In fact, a measurable set S possessing no content results if a positive integer, n, is placed into S if and only if the non-negative integer $k = k_n$ defined by $2^k \leqq n < 2^{k+1}$ is even. The existence of $L(S)$ and the non-existence of $M(S)$ for this S follow by straightforward counting.

A striking explanation of the difference between the existence of Dirichlet's and Frobenius' limits is contained in the remark that *the transition from measure to content is equivalent to the replacement of**

$$\sum \frac{S(n)}{n} r^n \sim \text{const.} \log \frac{1}{1 - r} \quad \text{by} \quad \sum S(n)r^n \sim \frac{\text{const.}}{1 - r},$$

i.e., to the differentiation of an asymptotic formula, as $r \to 1$. This may be seen as follows: If $\sum a(n)$ is any real series with non-negative terms the sum of the first n of which is $O(n^\epsilon)$ for every fixed $\epsilon > 0$, then not only (56) holds but also

$$\text{(56 bis)} \qquad \sum_{n=1}^{(1-r)^{-1}} a(n) \sim \sum_{n=1}^{\infty} a(n)r^n \qquad \text{as } r \to 1.$$

In fact, since $a(n) \geqq 0$, the $O(n^\epsilon)$-condition is identical with the condition of "slowness" (cf. §31) and so (56 bis) follows from the same source as (56); cf. Pringsheim [33] and Karamata [22]. On the other hand, the finite sums occurring in (56) and (56 bis) are identical if $\exp(1/s) = (1 - r)^{-1}$. Consequently, (56) and (56 bis) imply that *the asymptotic formula*

$$\text{(56*)} \qquad \sum a(n)n^{1/\log(1-r)} \sim \sum a(n)r^n \quad \text{as} \quad r \to 1$$

holds whenever $a(1) + \cdots + a(n) = O(n^\epsilon)$ *for every* $\epsilon > 0$ *and* $a(n) \geqq 0$ *for every* n. Since both of the latter assumptions are satisfied if $a(n) = S(n)/n$, where $0 \leqq S(n) \leqq 1$, the italicized statement preceding (56 bis) follows from (α^*) and (β^*).

* If const. $= 0$, these asymptotic formulae are to be interpreted as the corresponding o-estimates.

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